Space and time tradeoffs for the k shortest simple paths problem

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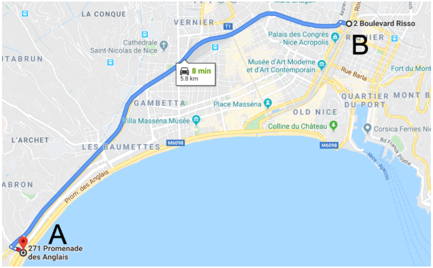
1. Introduction
2. k shortest simple paths problem
3. k shortest simple paths algorithms :
    - Yen's algorithm
    - Kurz and Mutzel's algorithm
4. Our contribution:
    - speeding up Kurz and Mutzel's algorithm
    - space time tradeoff

Motivation

A shortest path is not enough!

- A shortest path may be affected
- Some constraints can be added
    - Bounded delay, cost ...
    - A user may prefer the coast road ...
- User likes diversity!



Give the user a set of 'good' choices

Motivation

Sometimes, it is hard to specify constraints that a path should satisfy

Applications:

- bioinformatics: biological sequence alignment
- natural language processing
- list decoding
- parsing
- network routing
- many more ...

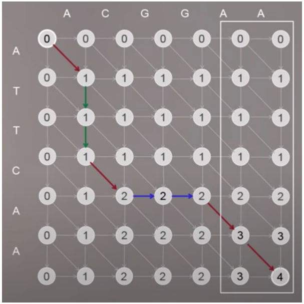


Figure: aligning two DNA sequences

k shortest paths problem

Definition

Input:

- Directed weighted graph $D = (V, A)$ with $w : A \to \mathbb{R}^+$,
- Two terminals s and t and an integer k

Output:

- k paths $P_1, P_2, \ldots, P_k$ from s to t such that $w(P_i) \leqslant w(P_{i+1})$, $1 \leqslant i < k$ and $w(P_k) \leqslant w(Q)$ for all other s-t paths Q

  where $w(P) = \sum_{e \in A(P)} w(e)$

simple vs not simple

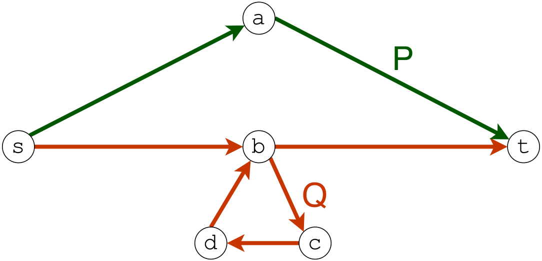


Figure: P is simple, Q is not simple

Definition (simple path)

a path is simple if and only if it has no repeated vertices

Complexity of the problem

Theorem (Eppstein '97)

The problem of finding k shortest paths can be solved in time $O(m + n \log n + k)$

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Theorem (Yen '71)

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Theorem (Yen '71)

The problem of finding k shortest simple paths can be solved in time $O(kn(m + n\log n))$

Theorem (Williams and Williams '10)

All-Pairs-Shortest-Paths (APSP) $<_{(m,n)}$ *2-SSP (* $\Leftrightarrow \tilde{O}(n.m)$ *for 2-SSP)*

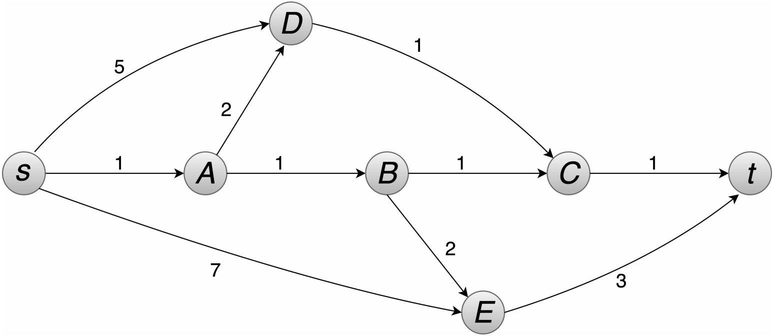
Yen's algorithm (the algorithm)

Yen's idea:

- A second shortest simple path is a shortest simple deviation from a shortest path

Complexity: $O(kn \qquad \underbrace{(m + nlogn)}_{\text{Complexity of finding one SP}} \qquad)$

Yen's algorithm (example)



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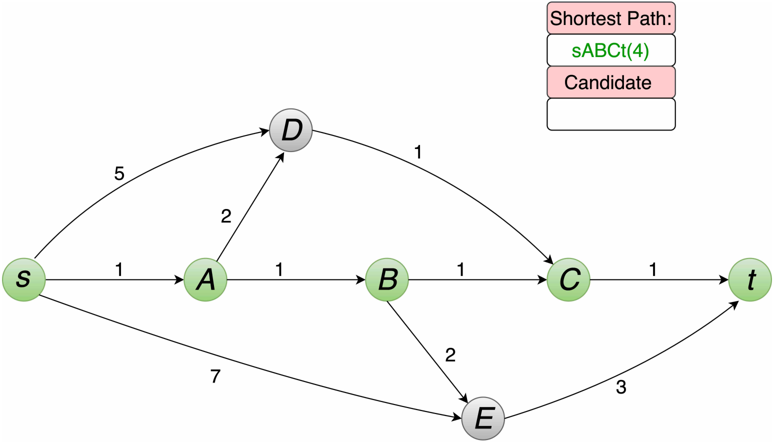

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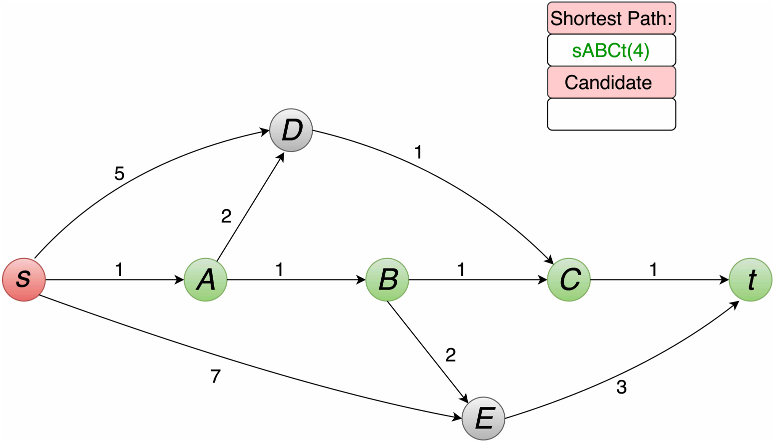

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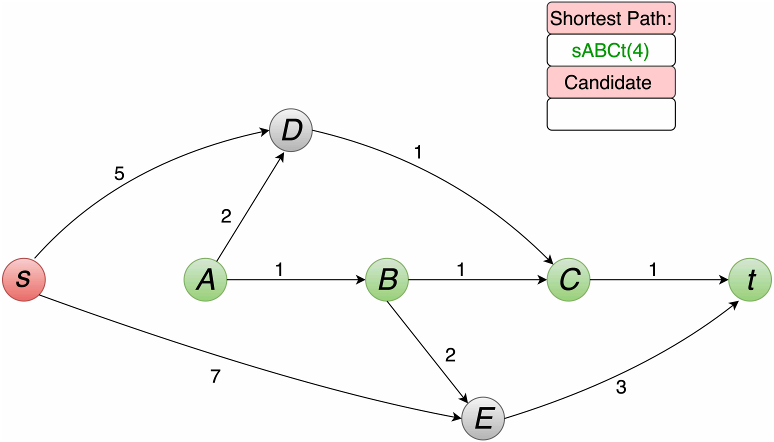

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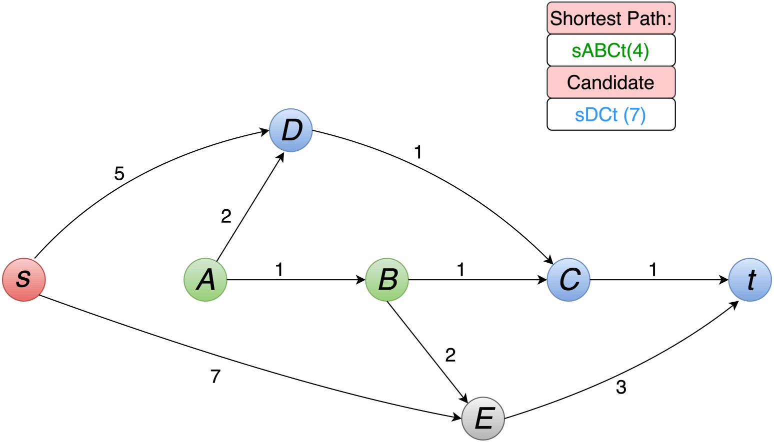

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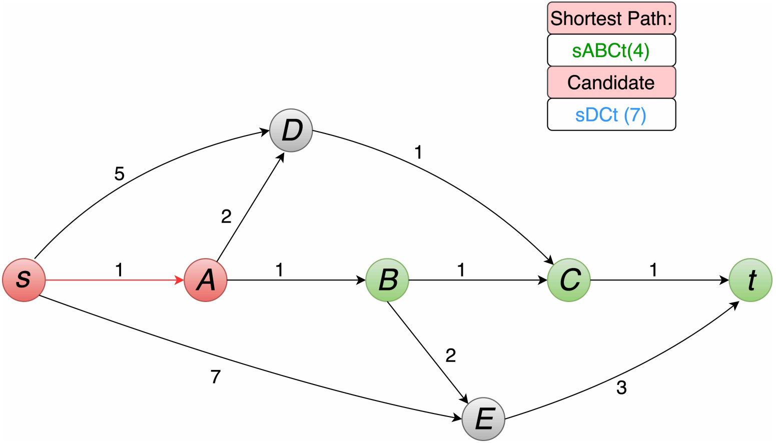

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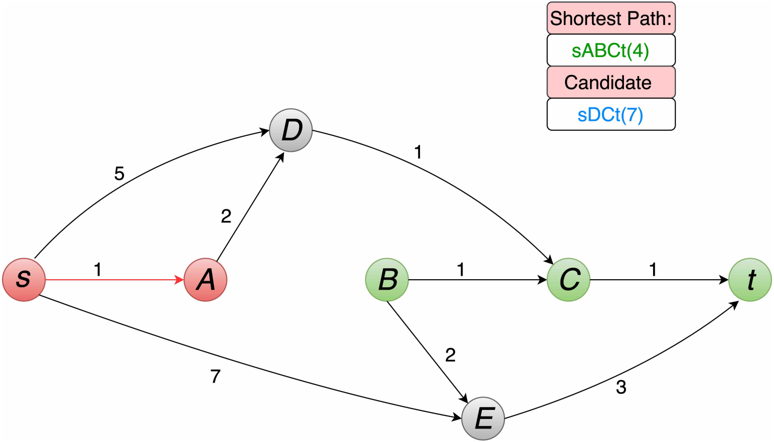

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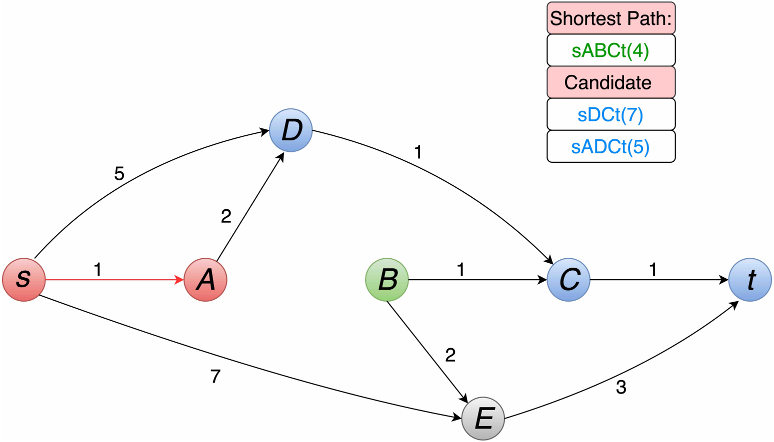

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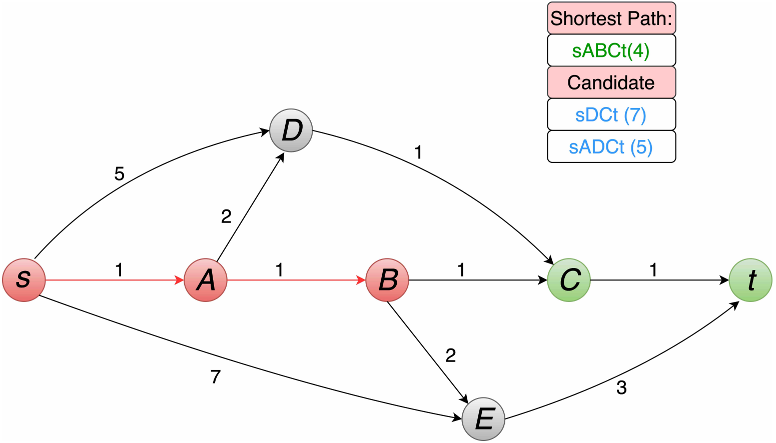

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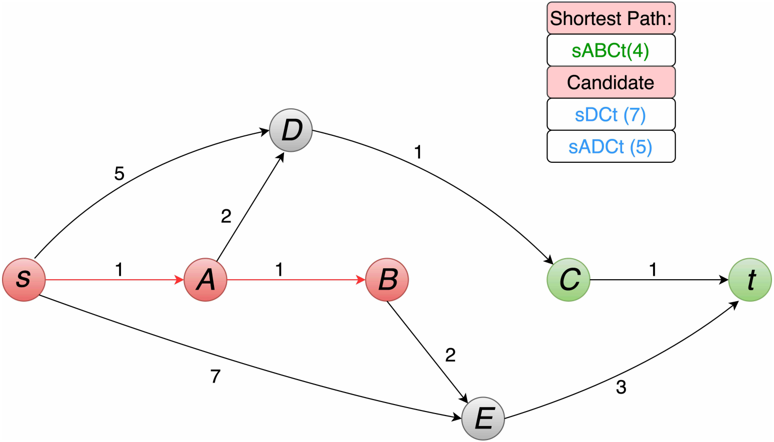

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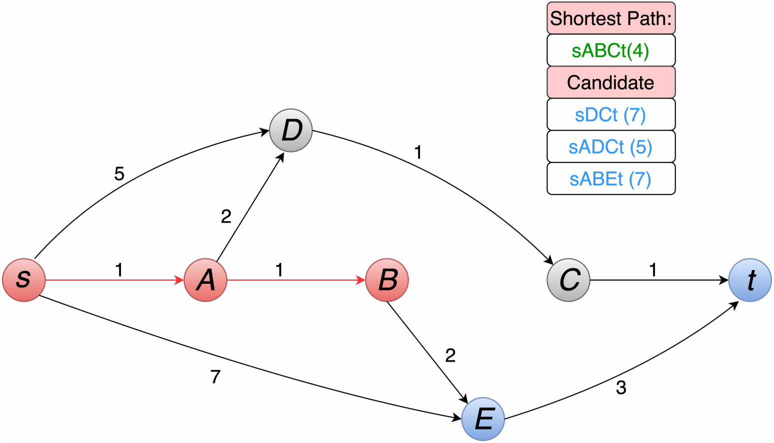

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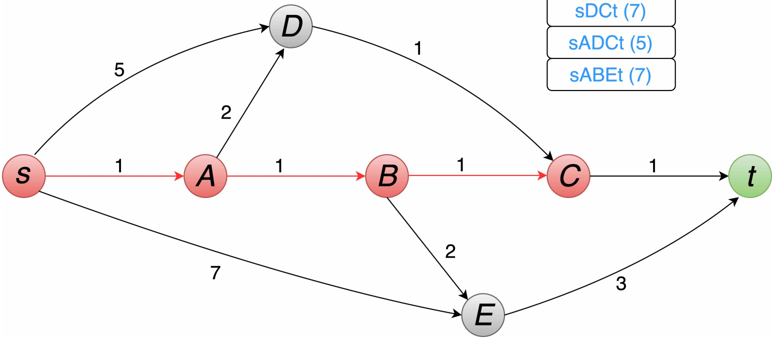
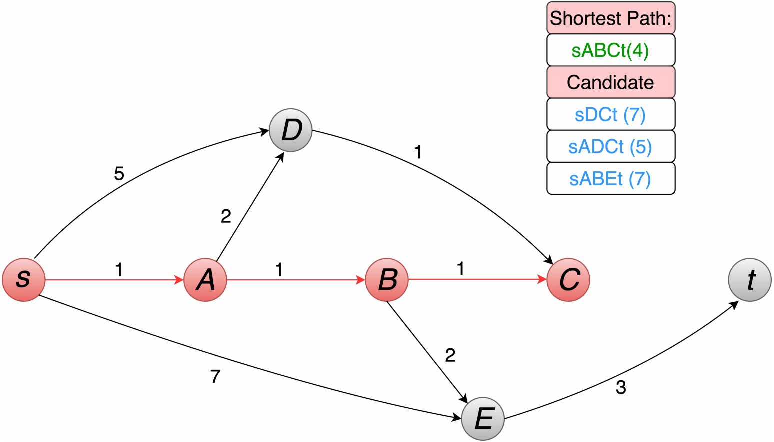


Shortest Path:
sABCt(4)
Candidate
sDCt (7)
sADCt (5)
sABEt (7)

Yen's algorithm (example)



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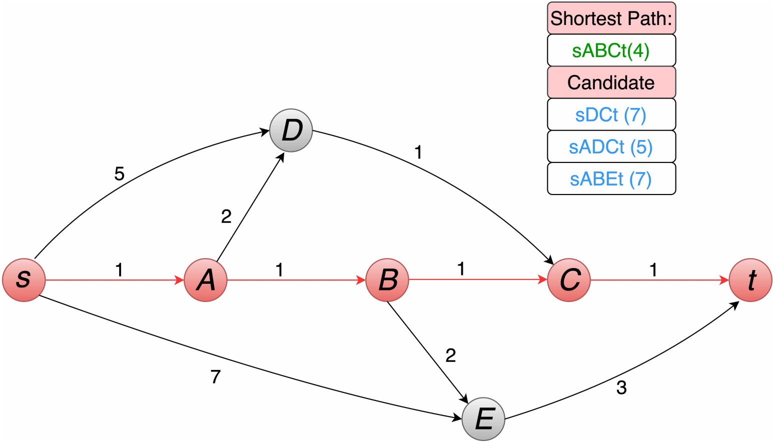

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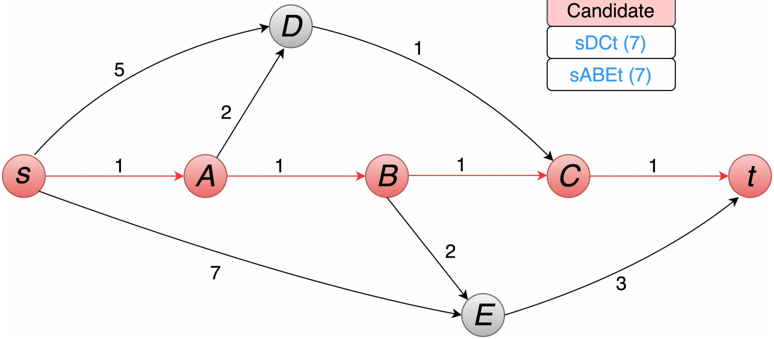

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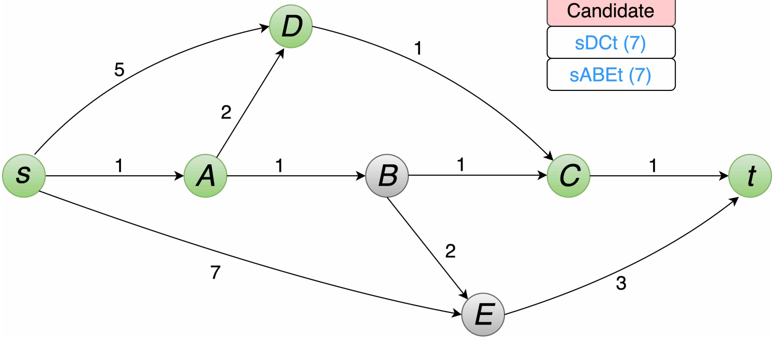

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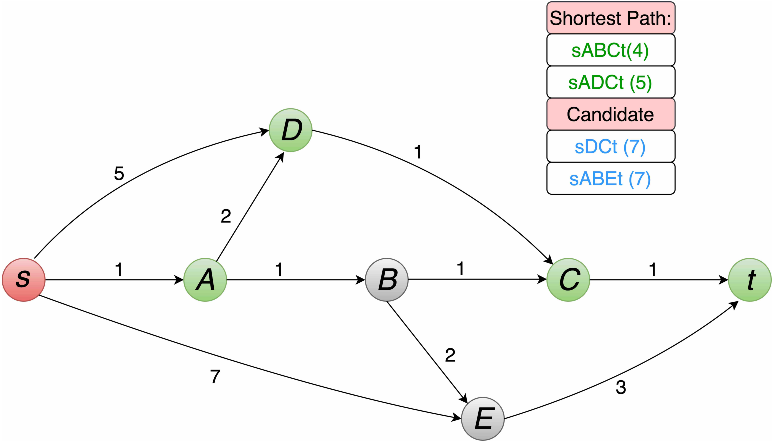

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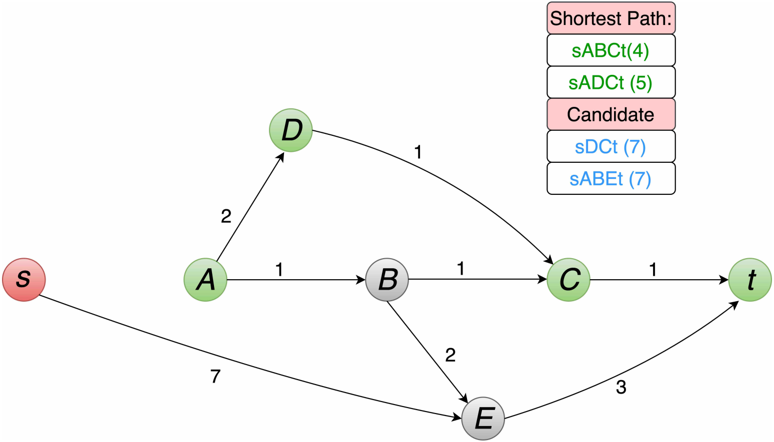

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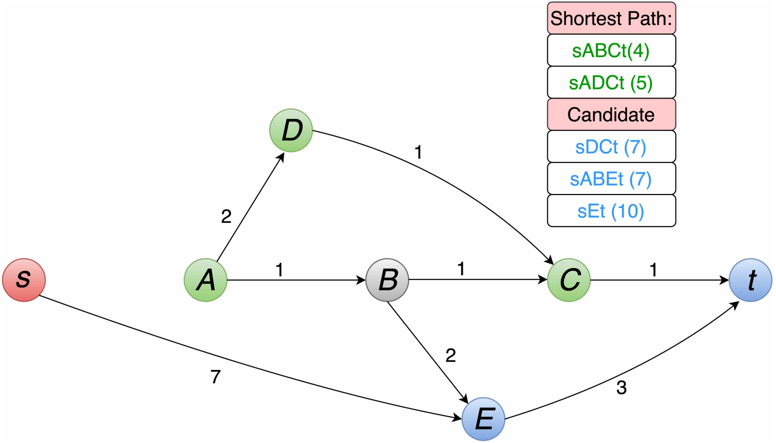

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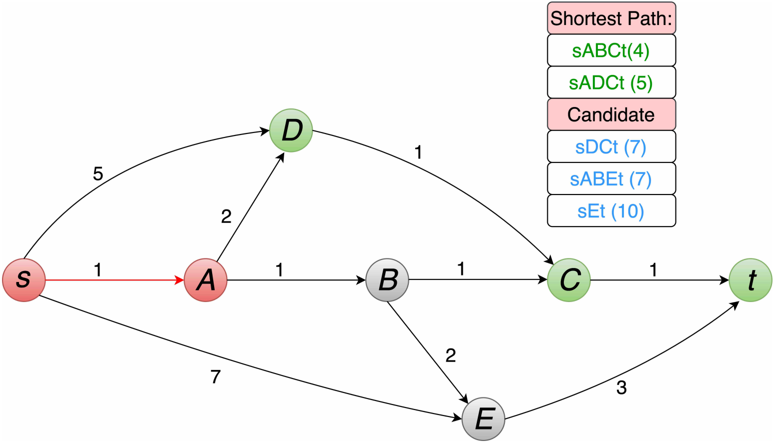

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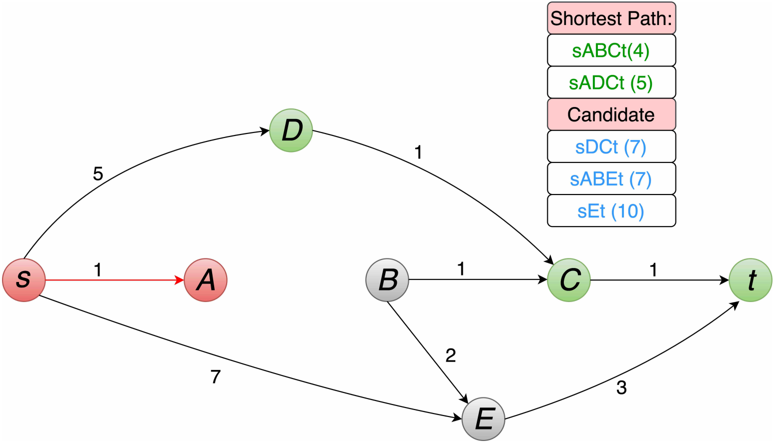

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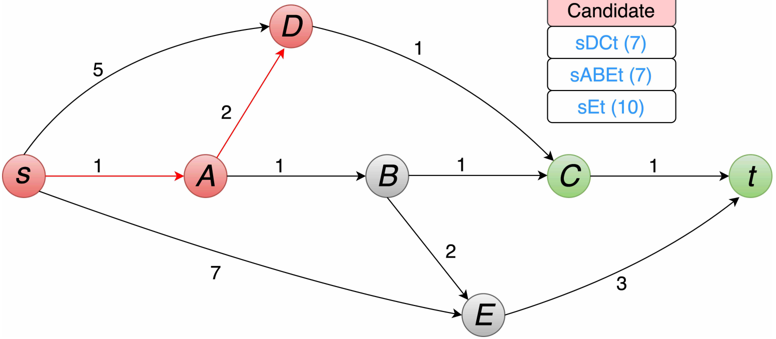

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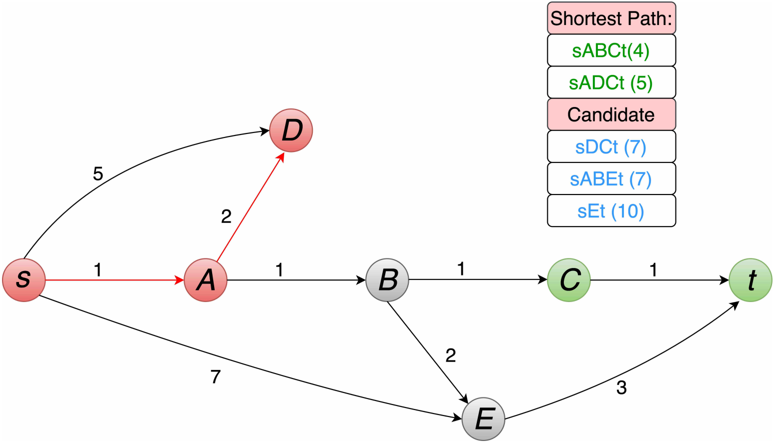

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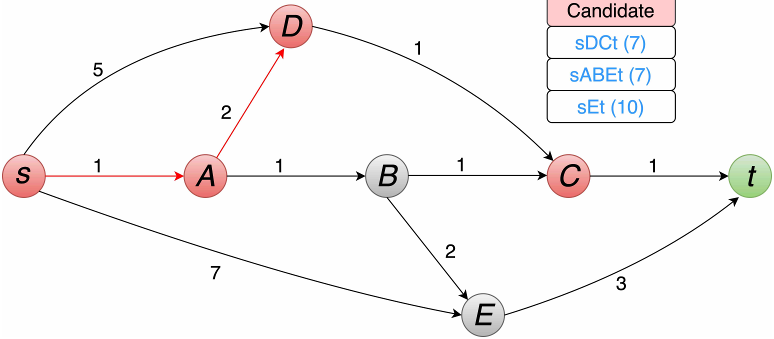

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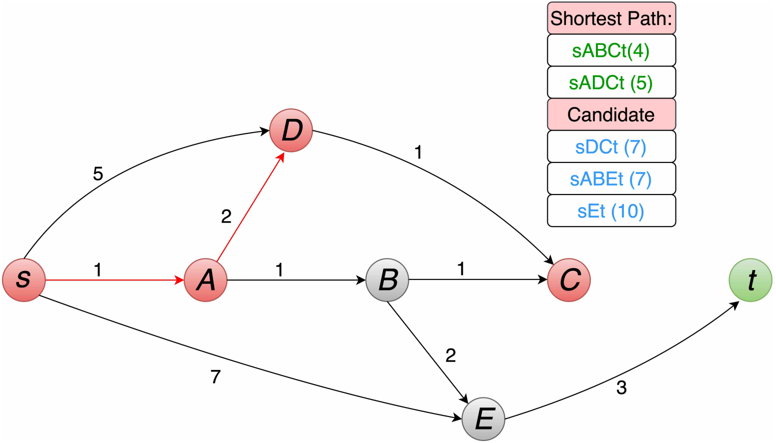

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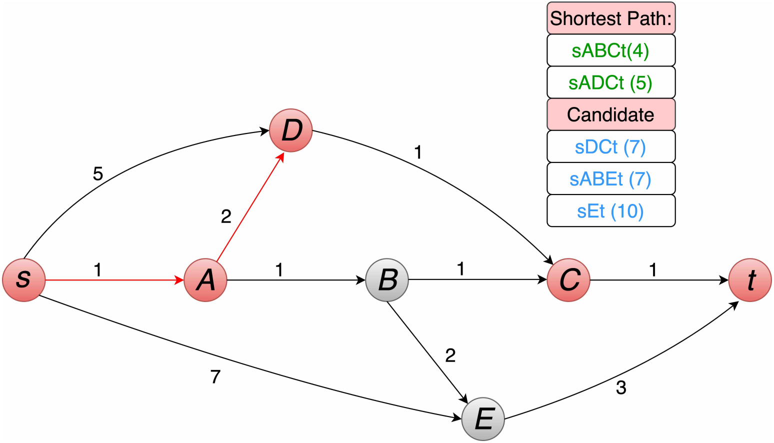

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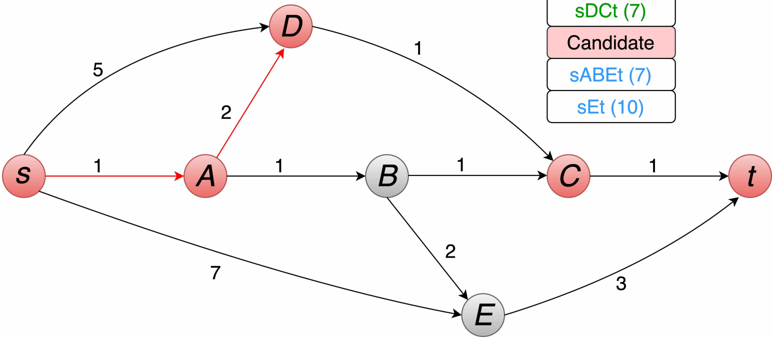

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Algorithm engineering

- 9th DIMAC'S implementation challenge followed by a set of improvements
- Feng 2014 (speed up the computation of deviations)
- Kurz and Mutzel 2016 (larger memory consumption)

Algorithm	time (s)
Yen	80
Feng	30
Kurz and Mutzel	1.15

Table: COL network ($n \approx 500,000$; $m \approx 1,000,000$ and $k = 300$)

- We proposed two improvements of Kurz and Mutzel's algorithm
  - Up to twice faster with the same memory consumption
  - A time-space trade-off

Kurz and Mutzel's algorithm (a path representation)

An *s*-*t* path can be represented as a sequence of arcs and shortest path trees

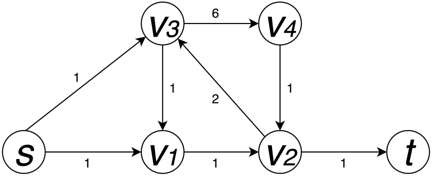


Figure: *G*

Kurz and Mutzel's algorithm (a path representation)

$P = \{s, v_3, v_1, v_2, v_3, v_4, v_2, t\}$ can be represented as $(T_0, e_1, T_0, e_2, T_1)$

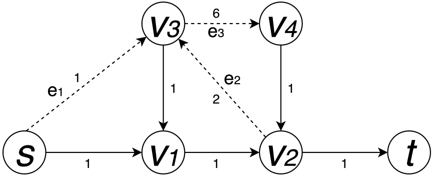


Figure: G

Note: P is not simple

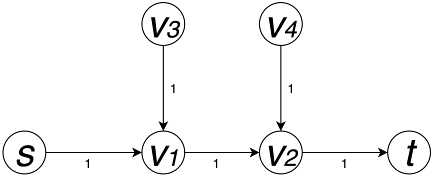


Figure: SP tree of G : T_0

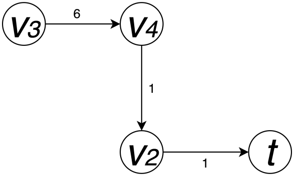


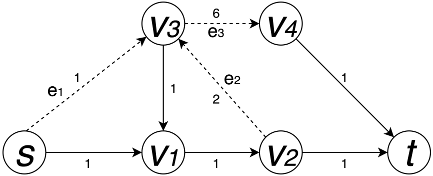
Figure: SP tree of $G \setminus \{S, v_1\}$: T_1

Kurz and Mutzel's algorithm (the algorithm)

Algorithm 1 $Kurz - Mutzel(G, s, t, k)$

1: $C \leftarrow \{(T_0)\}$ // where T_0 is a shortest path tree of G
2: **while** C is not empty **do**
3: $\quad P \leftarrow extractMin(C)$
4: $\quad$ **if** P is simple **then**
5: $\quad\quad$ add P to the output
6: $\quad\quad$ add the extensions of P to C
7: $\quad$ **else**
8: $\quad\quad$ **if** P can be "repaired" **then**
9: $\quad\quad\quad$ repair P into a simple path and add it to C

Kurz and Mutzel's algorithm (example)



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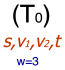


Figure: the heap C

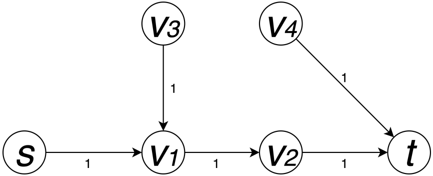
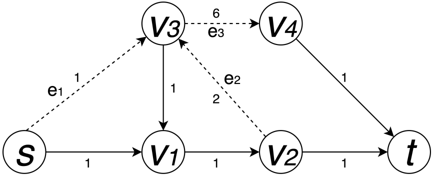


Figure: T_0

Output: s, v_1, v_2, t

Kurz and Mutzel's algorithm (example)



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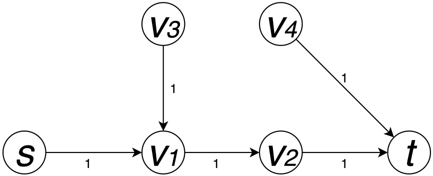

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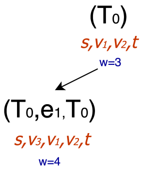

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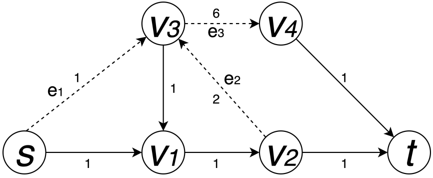
Kurz and Mutzel's algorithm (example)



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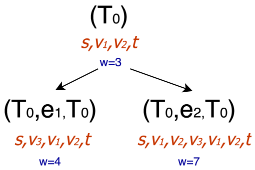

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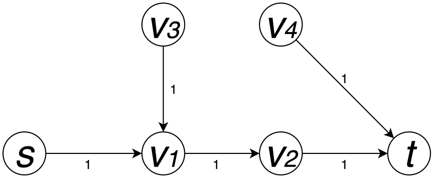

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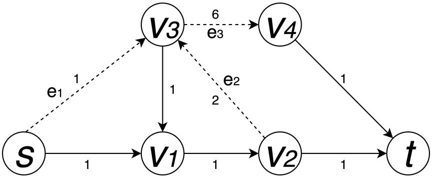

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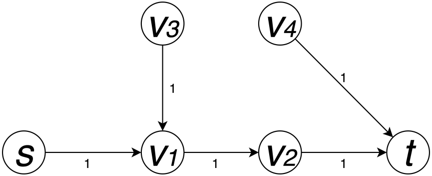

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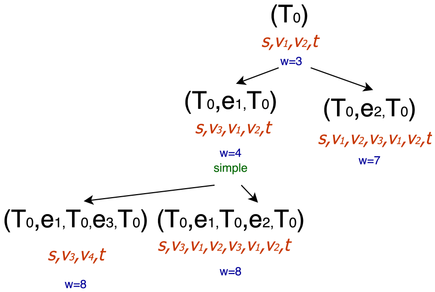

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s, v_3, v_1, v_2, t

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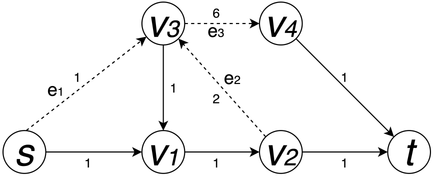

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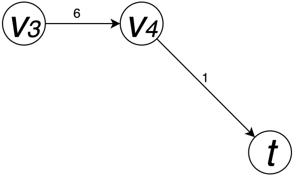


Figure: T_1

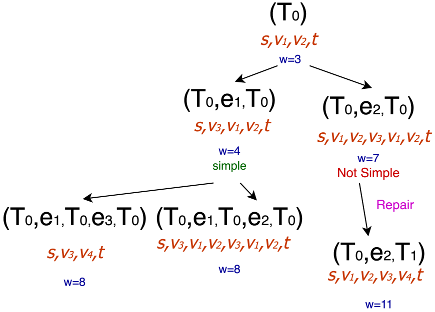


Figure: the heap C

Output: s, v_1, v_2, t

s, v_3, v_1, v_2, t

Kurz and Mutzel's ('16) improvements

- Verify if a path is simple or not in a pivot step
- Using a Lazy Dijkstra (stop once the path is constructed)
- Split the heap C into two (C_{simple} and $C_{not-simple}$)
- ...

Our first improvement

Once a no simple path $P = (T_0, e_0, \cdots, T_h, e_h, T_h)$ is extracted, the algorithm repair P ($\Rightarrow$ computing a new SP tree T')

- Remark: T' and T_h are "similar" and T_h is computed and stored
- Instead of computing T' from scratch
    - Compute T' starting from T_h

Our first improvement - evaluation

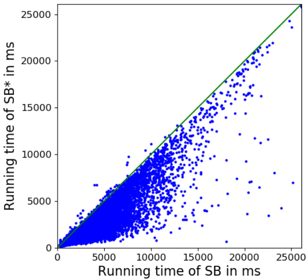


Figure: Rome ($n \approx 3000, m \approx 9000$ and $k = 10,000$)

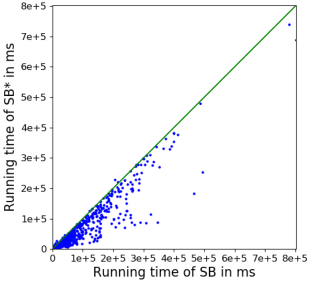


Figure: COL ($n \approx 500,000, m \approx 10^6$ and $k = 1000$)

Evaluation of the improvement on Dimac's routing networks

- A speed up by a factor of 1.5 to 2 on average

Publicly available: `https://gitlab.inria.fr/dcoudert/k-shortest-simple-paths`

Space-time trade off

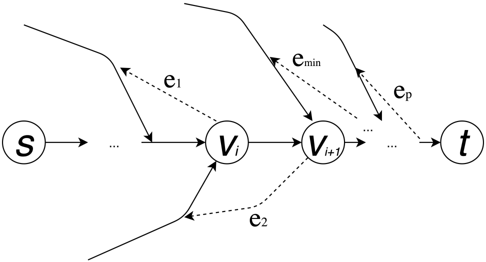
In practice, memory consumption is a BIG issue

- One shortest path tree of a graph of one million vertices needs $\approx 1MB$
- For big values of k, a large number of shortest path trees should be stored
- One may use Feng's algorithm, but it is too slow.

Goal: space time tradeoff

Space-time trade off

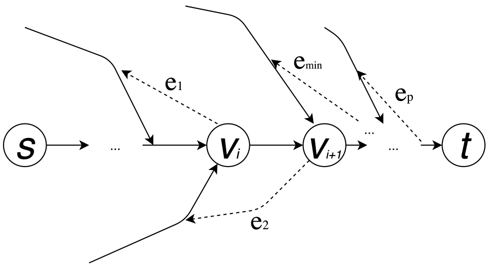
Let $P = (s, v_1, \cdots, v_i, \cdots, t)$ be a path extracted from C, and let $E = \{e_1, \cdots, e_{min}, \cdots, e_p\}$ be the set of deviations tailing at P and leading to a no simple extension, with e_{min} the deviation with the smallest weight lower bound



Kurz-Mutzel's algorithm:

- May computes independently a new tree for each vertex with a deviation tailing at it
- Each tree remains in the memory till the end of the execution

Space-time trade off

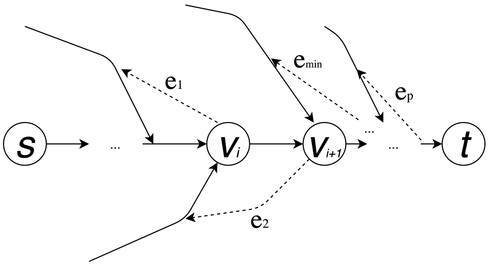


Remark : All these trees are "similar"

The improvement:

- Compute simultaneously the simple extensions at $e_{min}, \cdots, e_p$
- Add them to C with their real cost as a key
- Add the extensions at $e_1, \cdots, e_{min-1}$ to C with the weight of the deviation at e_{min} as a key
- Only the tree of the extensions at e_{min} is saved

Space-time trade off



Consequences:

- All of these no simple extension after e_{min} have a higher key in C and their extraction could be (hopefully) skipped
- Their corresponding trees are freed from the memory
- Unfortunately, some trees may be re-computed

Space-time trade off - evaluation

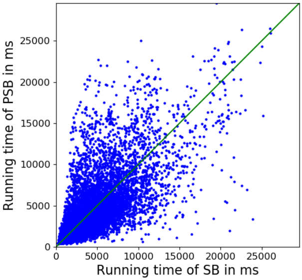


Figure: Running time

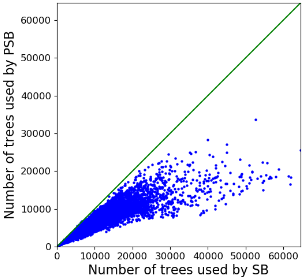


Figure: # stored trees

Evaluation of the improvement on the routing network of Rome ($n \approx 3000, m \approx 9000$ and $k = 10,000$)

- A space reduction by a factor of 1.5 to 2 on average with a comparable running time

Space-time trade off - future work

The goal:

- A tree is stored in the memory if and only if it will be used during the execution of the algorithm
    - No trees re-computation
    - Less space consuming
- Remark: Only trees used for (relatively) shortest paths are re-used
    - Store a tree only if it is used to extend a path that is shorter than a threshold value
- Analyse other parameters (number of hops of paths, size of the input graph) in order to know which algorithm is better for each input

Questions ?