Regret-equality in Stable Marriage

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Outline

• Matching problems

• Fairness

• Finding fair stable matchings

• Experiments

• Future work
Matching Problems

• Assign one set of entities to another set of entities

• Based on preferences and capacities
Stable Marriage

Cost: $c_U(M) = 10$, $c_W(M) = 10$

Degree: $d_U(M) = 4$, $d_W(M) = 4$

Blocking pair

A stable matching is a matching with no blocking pairs
Stable Marriage

- A **stable matching** is a matching with no blocking pairs
- Many stable matchings per instance
- We can find a stable matching in linear time using the man-oriented or woman-oriented Gale-Shapley Algorithm. $O(m)$ time where $m$ is total length of preference lists
- Man-oriented Gale-Shapley Algorithm: finds a man-optimal (woman-pessimal) stable matching (and vice versa)
Fairness

- Want to find a stable matching that provides some kind of equality between men and women

- Several different fairness measures
Fairness measures

Among all stable matchings, find the stable matching that…

<table>
<thead>
<tr>
<th>Cost: $c_U(M), c_W(M)$</th>
<th>Degree: $d_U(M), d_W(M)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>balanced score</td>
<td>degree</td>
</tr>
<tr>
<td>Balanced stable matching NP-hard</td>
<td>Minimum-regret stable matching Poly</td>
</tr>
<tr>
<td>sex-equal score</td>
<td>regret-equal score</td>
</tr>
<tr>
<td>Sex-equal stable matching NP-hard</td>
<td>* Regret-equal stable matching ?</td>
</tr>
<tr>
<td>egalitarian cost</td>
<td>regret sum score</td>
</tr>
<tr>
<td>Egalitarian stable matching Poly</td>
<td>* Min-regret sum stable matching ?</td>
</tr>
</tbody>
</table>
Fairness measures (degree based)

10 stable matchings for this instance

\[
\begin{align*}
  m_1: & \ w_1, w_2, \underline{w_3}, w_4 \\
  m_2: & \ w_2, \underline{w_1}, w_4, w_3 \\
  m_3: & \ w_3, \underline{w_4}, w_1, w_2 \\
  m_4: & \ w_4, \underline{w_3}, w_2, w_1 \\
  m_1: & \ w_1, w_2, \underline{w_3}, w_4 \\
  m_2: & \ w_2, \underline{w_1}, w_4, w_3 \\
  m_3: & \ w_3, \underline{w_4}, w_1, w_2 \\
  m_4: & \ w_4, \underline{w_3}, w_2, w_1 \\
  m_1: & \ w_1, w_2, w_3, w_4 \\
  m_2: & \ w_2, \underline{w_1}, w_4, w_3 \\
  m_3: & \ w_3, \underline{w_4}, w_1, w_2 \\
  m_4: & \ w_4, \underline{w_3}, w_2, w_1 \\
  m_1: & \ w_4, m_3, m_2, m_1 \\
  m_2: & \ m_3, m_4, m_2, m_1 \\
  m_3: & \ m_2, m_1, m_4, m_3 \\
  m_4: & \ m_1, m_2, m_3, m_4 \\
  m_1: & \ w_1, w_2, \underline{w_3}, w_4 \\
  m_2: & \ w_2, \underline{w_1}, w_4, w_3 \\
  m_3: & \ w_3, \underline{w_4}, w_1, w_2 \\
  m_4: & \ w_4, \underline{w_3}, w_2, w_1 \\
  m_1: & \ w_4, m_3, m_2, m_1 \\
  m_2: & \ m_3, m_4, m_2, m_1 \\
  m_3: & \ m_2, m_1, m_4, m_3 \\
  m_4: & \ m_1, m_2, m_3, m_4 \\
  m_1: & \ w_4, \underline{w_3}, w_2, w_1 \\
  m_2: & \ m_3, m_4, \underline{m_2}, \underline{m_1} \\
  m_3: & \ m_2, m_1, m_4, m_3 \\
  m_4: & \ m_1, m_2, m_3, m_4 \\
\end{align*}
\]

- **Min-regret & Regret-equal**
  - Degree: 3
  - Regret-equality score: 0
  - Min-regret sum score: 6

- **Min-regret & Min-regret sum**
  - Degree: 3
  - Regret-equality score: 1
  - Min-regret sum score: 5

- **Min-regret sum**
  - Degree: 4
  - Regret-equality score: 3
  - Min-regret sum score: 5

Over all stable matchings:
- Minimum degree = 3
- Minimum regret-equality score = 0
- Minimum regret sum score = 5
Finding a Regret-Equal Stable Matching
Rotations

- Rotation - series of man-woman pairs that take us from one stable matching to another when permuted

  \[ R_1 \]
  \[ m_1 \ m_4 \]
  \[ w_2 \ w_3 \]

- Can only eliminate exposed rotations

  \[ R_2 \]
  \[ m_1 \ m_2 \]
  \[ w_1 \ w_2 \]

- O(n^2) algorithm to find all rotations

- Rotations form a structure to allow enumeration of all stable matchings. All rotation makes some men worse off and some women better off
Algorithm

1. Find the man-optimal stable matching $M_0$

   - Each man has their best partner in any stable matching. Say $d_u(M_0) = 2$ and $d_w(M_0) = 5$  
     $d(M_0) = (2, 5)$

   - Then, a regret equal stable matching must exist within the following degrees pairs:

     - $M_0$ has a r-e score of 3
     - men can only get worse
     - women can only get better

     - why are these the only possible degrees?

     - r-e score: 3 \( (2, 5) \)
     - r-e score: 2 \( (2, 4) \) \( (3, 5) \)
     - r-e score: 1 \( (2, 3) \) \( (3, 4) \) \( (4, 5) \)
     - r-e score: 0 \( (2, 2) \) \( (3, 3) \) \( (4, 4) \) \( (5, 5) \)
     - r-e score: 1 \( (2, 1) \) \( (3, 2) \) \( (4, 3) \) \( (5, 4) \) \( (6, 5) \)
     - r-e score: 2 \( (3, 1) \) \( (4, 2) \) \( (5, 3) \) \( (6, 4) \) \( (7, 5) \)
Algorithm

2. If $d_U(M_0) \geq d_W(M_0)$ then exit with $M_0$

3. For each man $m$ and for each column $c$:
   1. rotate $m$ down to $c$ (if possible)
   2. rotate women down column $c$ who have worst rank

   r-e score: 3  \[(2, 5)\]
   r-e score: 2  \[(2, 4) (3, 5)\]
   r-e score: 1  \[(2, 3) (3, 4) (4, 5)\]
   r-e score: 0  \[(2, 2) (3, 3) (4, 4) (5, 5)\]
   r-e score: 1  \[(2, 1) (3, 2) (4, 3) (5, 4) (6, 5)\]
   r-e score: 2  \[(3, 1) (4, 2) (5, 3) (6, 4) (7, 5)\]

   • Stop iterating women up the column when $d_U(M) \geq d_W(M)$
   • Save the best matching as you go
Time complexity

- Find man-optimal stable matching & all rotations \( O(m) \)
- For each man \( O(n) \)

\[
2 \times \text{man-optimal difference}
\]
- For each column \( O(2 \times |d_u(M_0) - d_w(M_0)|) = O(c) \)
- Rotate man up and women down \( O(m) \)

Total \( O(nmc) \)
Experiments
Methodology

• Performance of the Regret-equal Algorithm compared to an Enumeration algorithm (exponential in worst case)

• Instances size \{10, 20, \ldots, 100, 200, \ldots, 1000\}, complete preference lists, 500 instance per size.

• looked at properties over several types of optimal stable matching (balanced, sex-equal, egalitarian, minimum regret, regret-equal, min-regret sum)

• Java, Python, Bash, GNU parallel

• Correctness
  
  • all matchings found were stable
  
  • Regret-equality scores matched
  
  • CPLEX up to size n = 50 for the enumeration algorithm
Time taken

<table>
<thead>
<tr>
<th>Mean time (ms)</th>
<th>100</th>
<th>1000</th>
<th>10000</th>
<th>100000</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>0</td>
<td>100</td>
<td>200</td>
<td>300</td>
</tr>
<tr>
<td>Enumeration Algorithm</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Regret-equal Algorithm</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

n

Frances Cooper
Regret-equality score for different optimal matchings

- Balanced
- Sex-equal
- Egalitarian
- Minimum regret
- Regret-equal
- Min-regret sum
Sex-equal score for different optimal matchings

- Balanced
- Sex-equal
- Egalitarian
- Minimum regret
- Regret-equal
- Min-regret sum
- Regret-Equal Algorithm

Mean sex-equal score vs. n
Frequency of different optimal stable matchings

- Balanced
- Sex-equal
- Egalitarian
- Minimum regret
- Regret-equal
- Min-regret sum

Mean number of stable matchings

- 100
- 400
- 700
- 1000
Future Work

• Improving the O(nmc) Regret-equal Algorithm, where $c = |d_U(M_0) - d_W(M_0)|$

• Grouping women - e.g. women are workers and men are jobs to assign to workers.
  
  • Woman optimal stable matching would naturally satisfy ‘balanced’, ‘min-regret’, ‘egalitarian’ and ‘min-regret sum’ criteria
  
  • Can find a ‘regret-equal’ stable matching in $O(n^4)$ time
  
  • Open problem for ‘sex-equality’ -> grouped-women-equality
Thank you

Summary

• Matching problems
• Fairness
• Finding fair stable matchings
• Experiments
• Future work: finding improved algorithms

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