Karlsruhe Institute of Technology

Advanced Flow-Based Multilevel Hypergraph Partitioning

SEA 2020

June 5, 2020

Lars Gottesbüren, Michael Hamann, Sebastian Schlag, Dorothea Wagner

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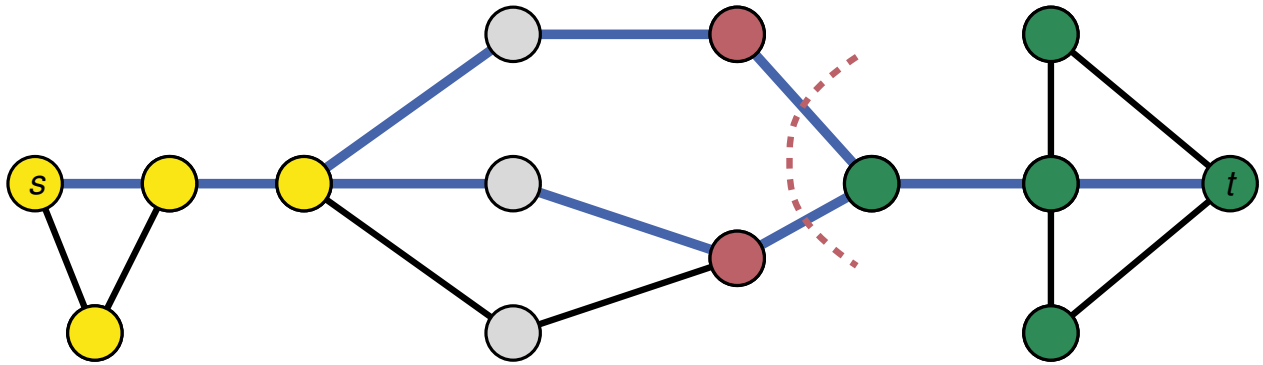
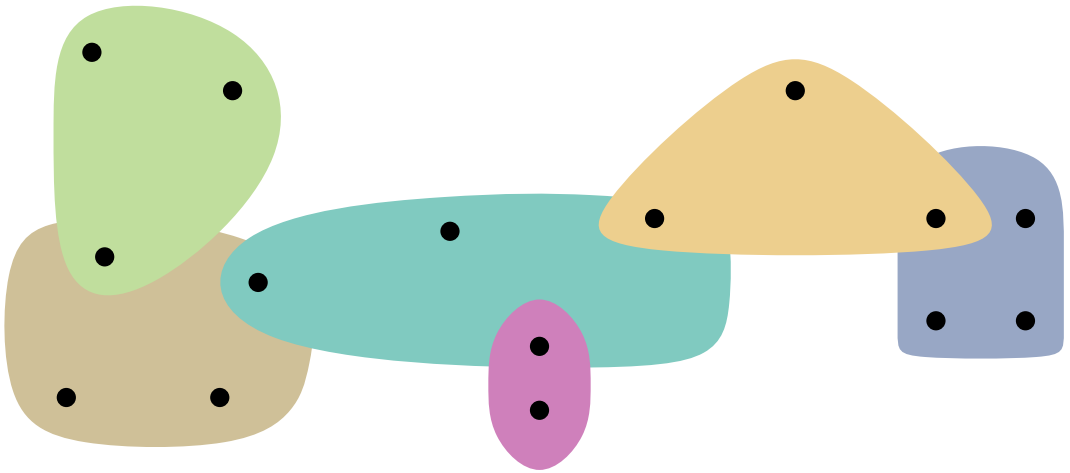


Hypergraph Partitioning

- Hypergraph $H = (V, E, c, \omega)$
  - vertex set $V = \{1, ..., n\}$
  - edge set $E \subseteq \mathcal{P}(V) \setminus \emptyset$
  - incident edges $\Gamma(u) = \{e \in E \mid u \in e\}$
  - vertex weights $\varphi : V \to \mathbb{R}_{\geq 1}$
  - edge weights $\omega : E \to \mathbb{R}_{\geq 1}$



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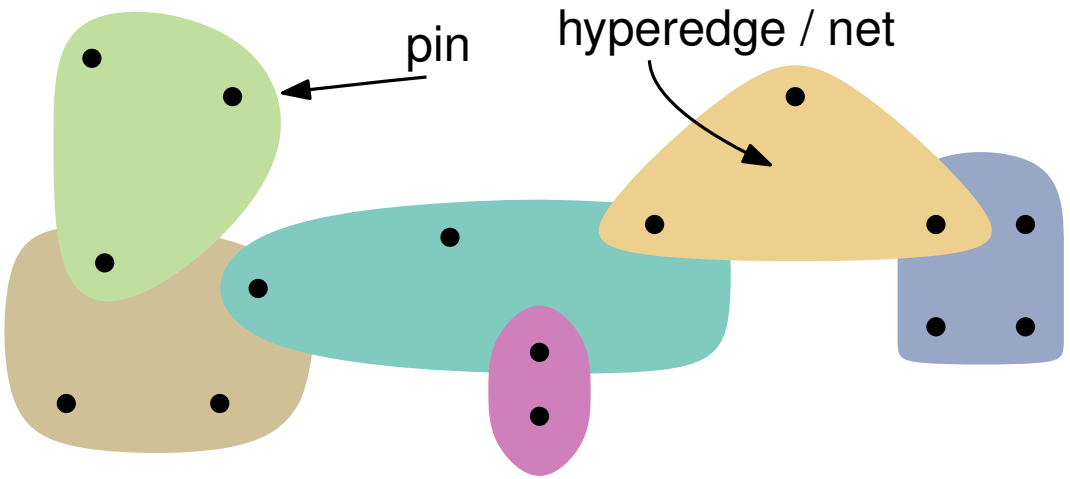

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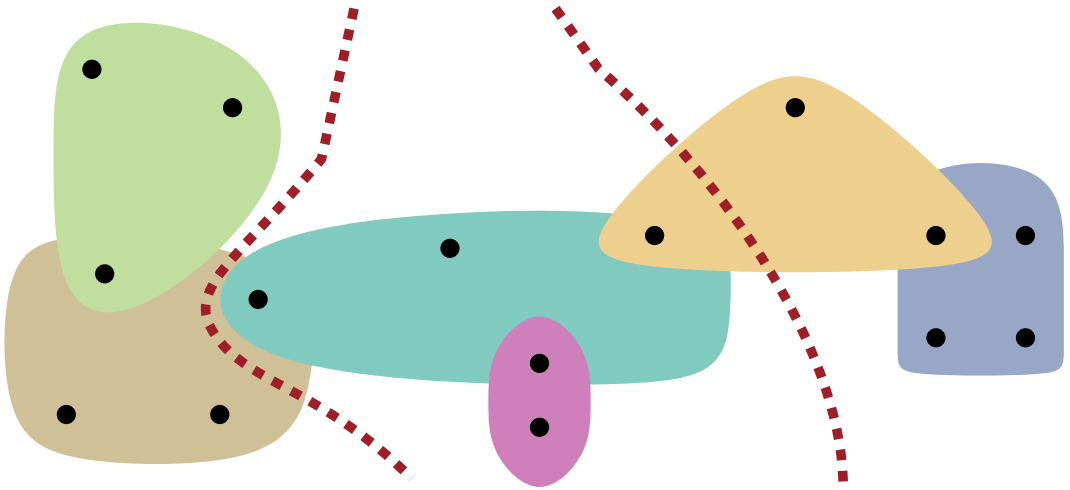
Hypergraph Partitioning

Partition into $\mathbf{k}$ disjoint blocks $\Pi = \{V_1, \ldots, V_k\}$

- blocks V_i have **roughly equal weight**:

$$\varphi(V_i) \leq (1 + \varepsilon) \left\lceil \frac{\varphi(V)}{k} \right\rceil$$



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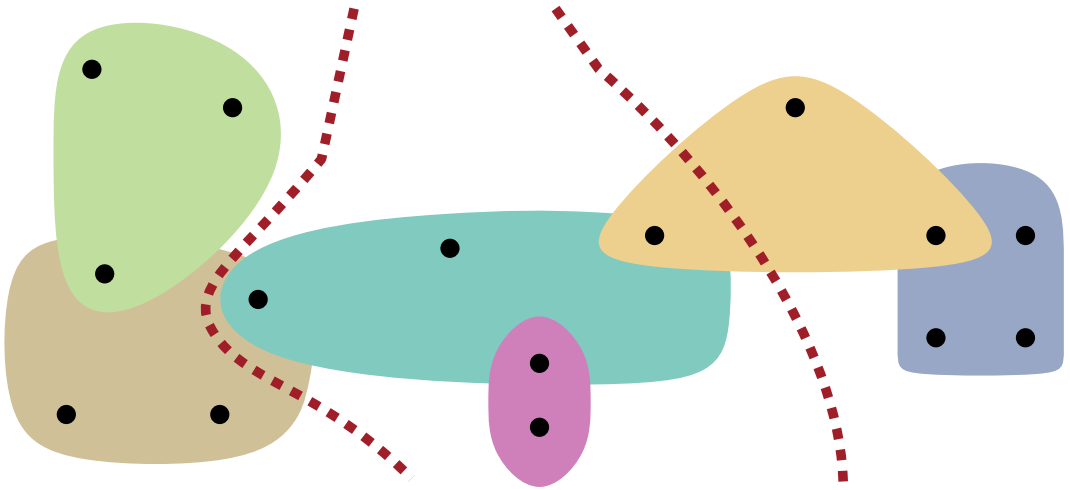

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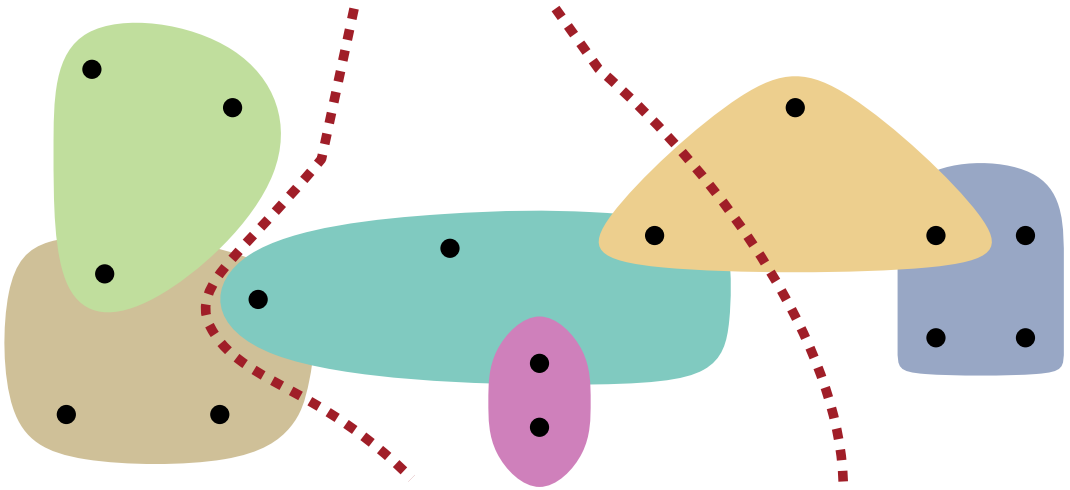
- **minimize connectivity** objective:

$$\text{con} = \sum_{e \in E} (\lambda(e) - 1)\, \omega(e)$$



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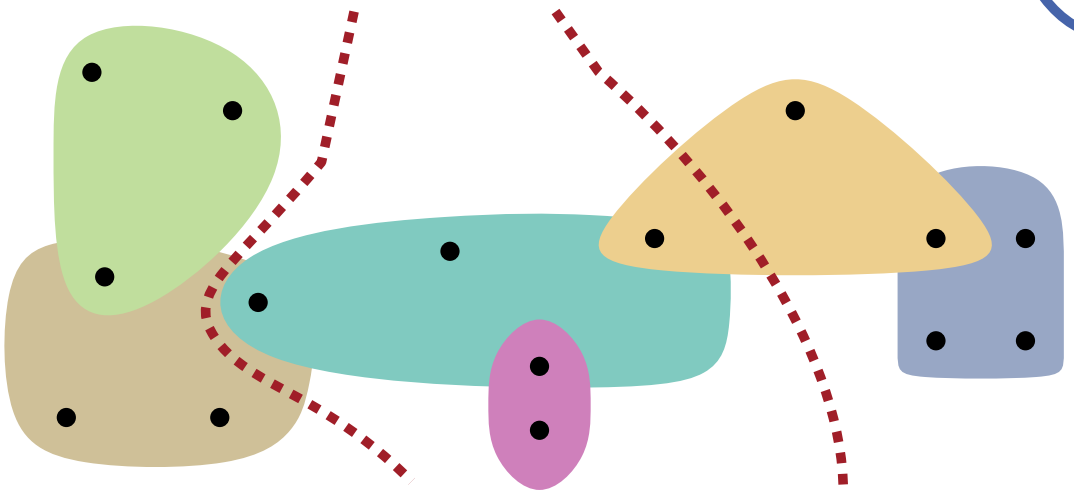

$\lambda(e) = |\{V_i \mid V_i \cap e \neq \emptyset\}|$
blocks overlapping with e

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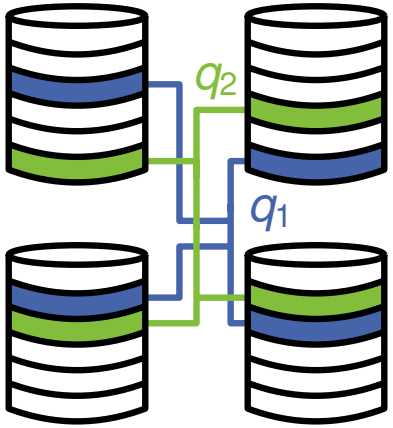
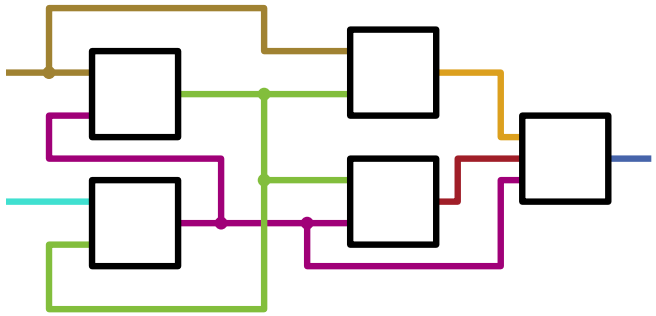
Applications



Distributed Databases

Route Planning



VLSI Design

HPC

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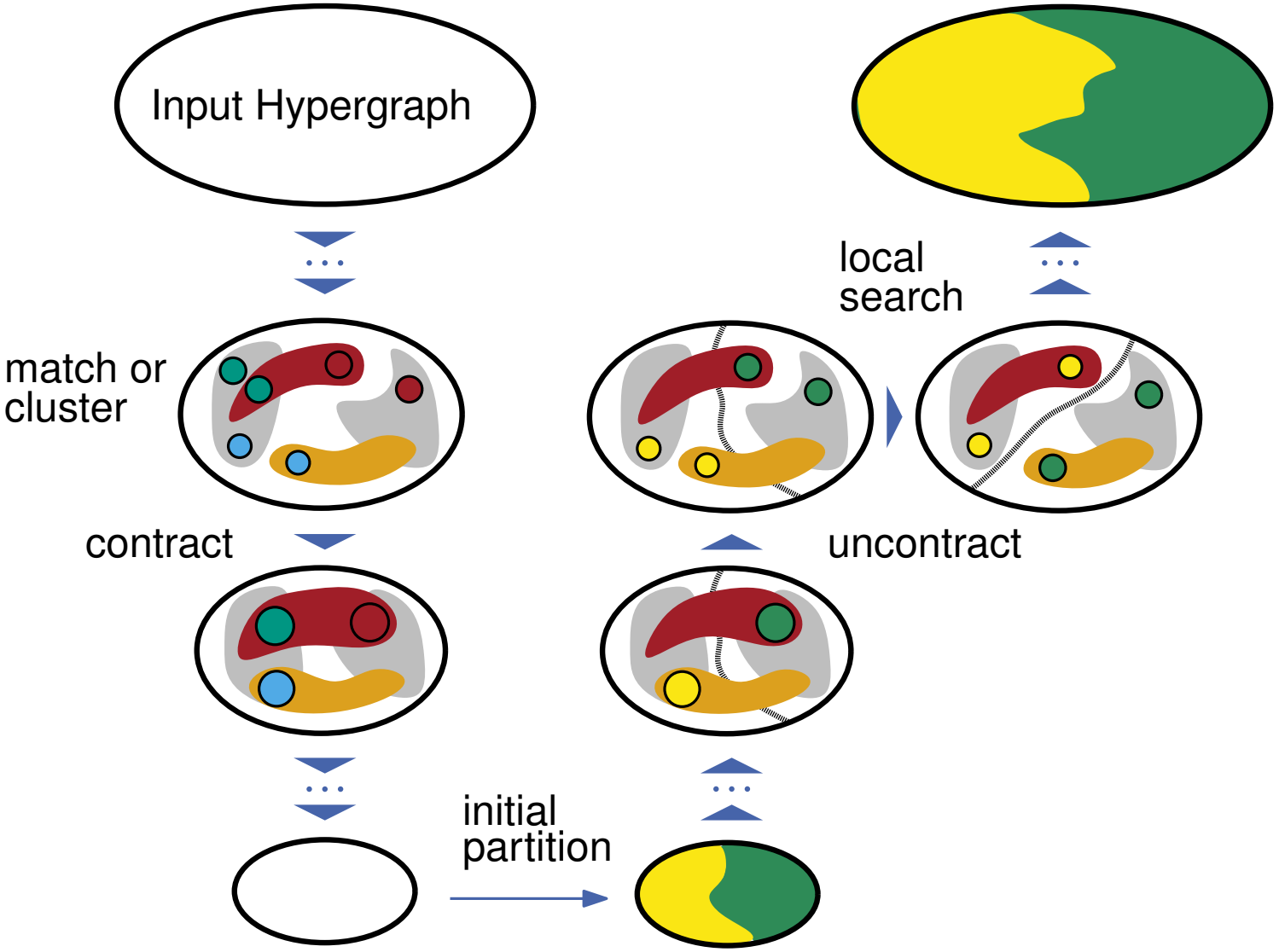
Multilevel Algorithms



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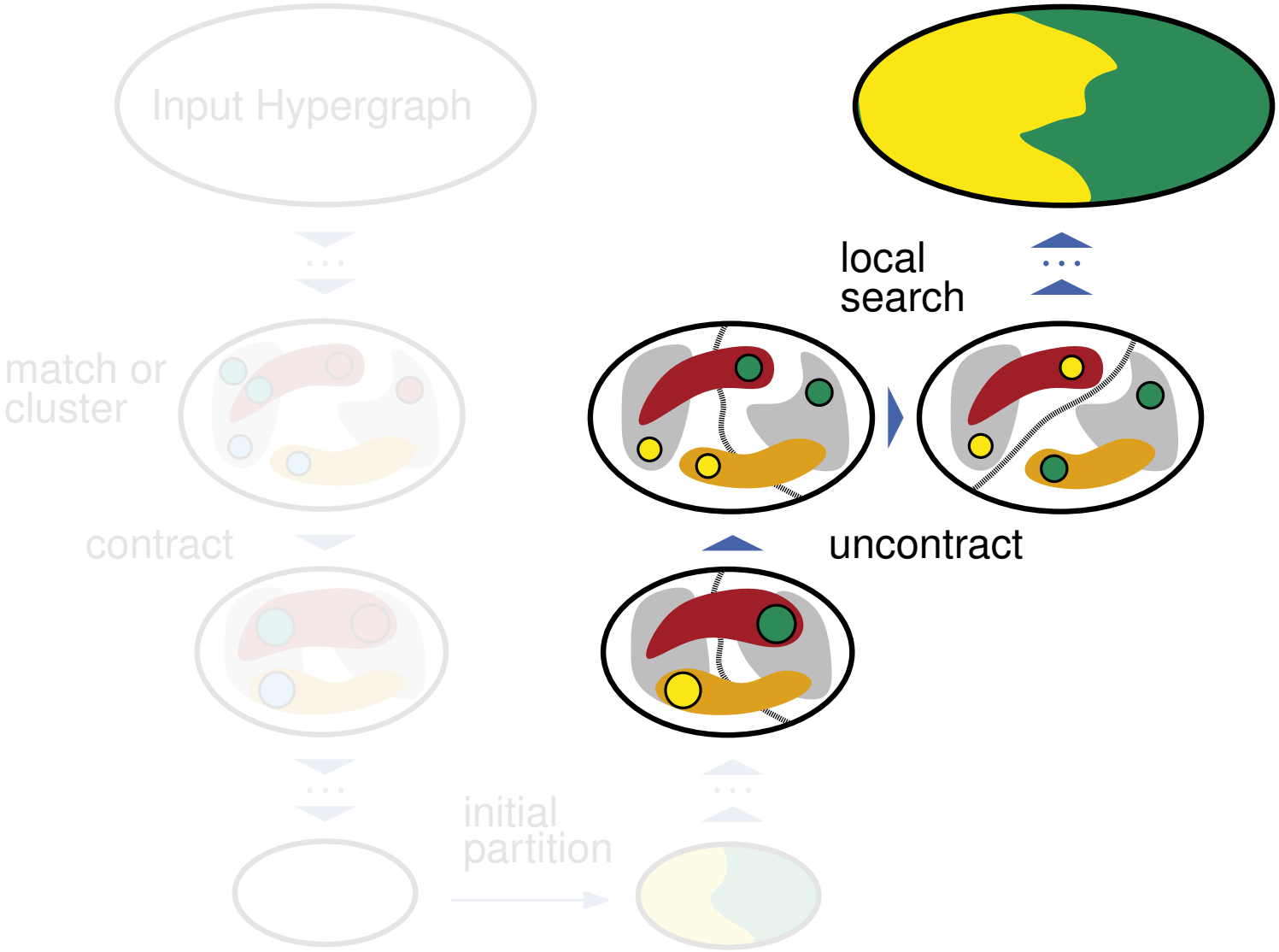
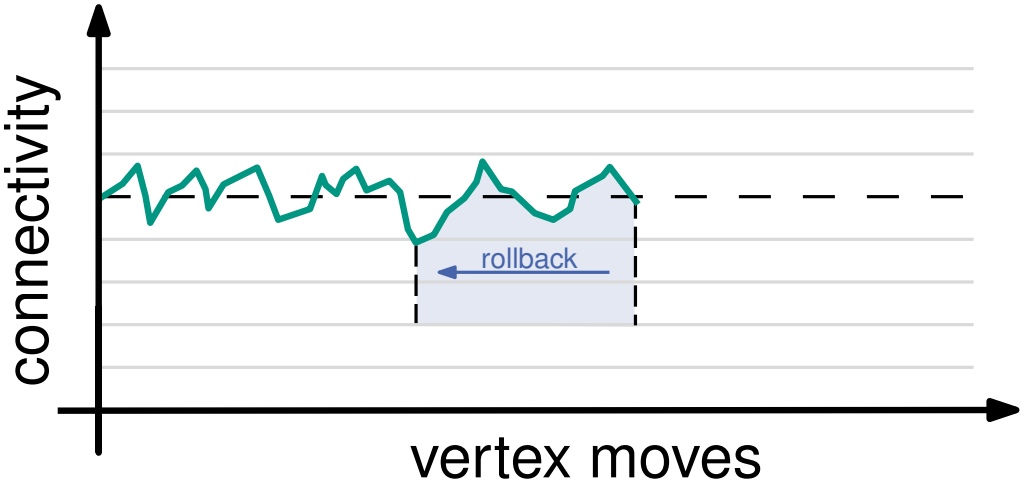


Classic Fiduccia-Mattheyses

Algorithm 1: FM Local Search

while *improvement found* **do**
  while ¬ *done* **do**
    `find` best move
    `perform` best move
  `rollback` to best solution

pass



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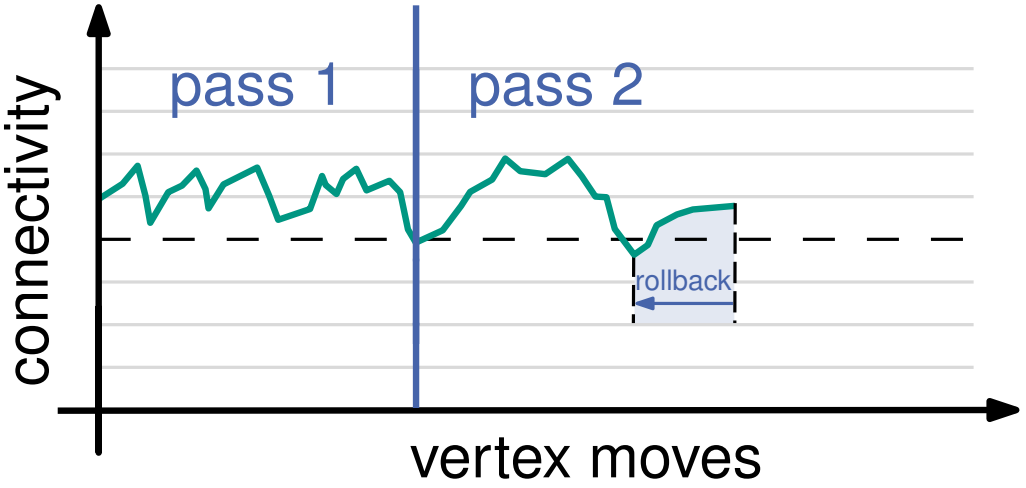

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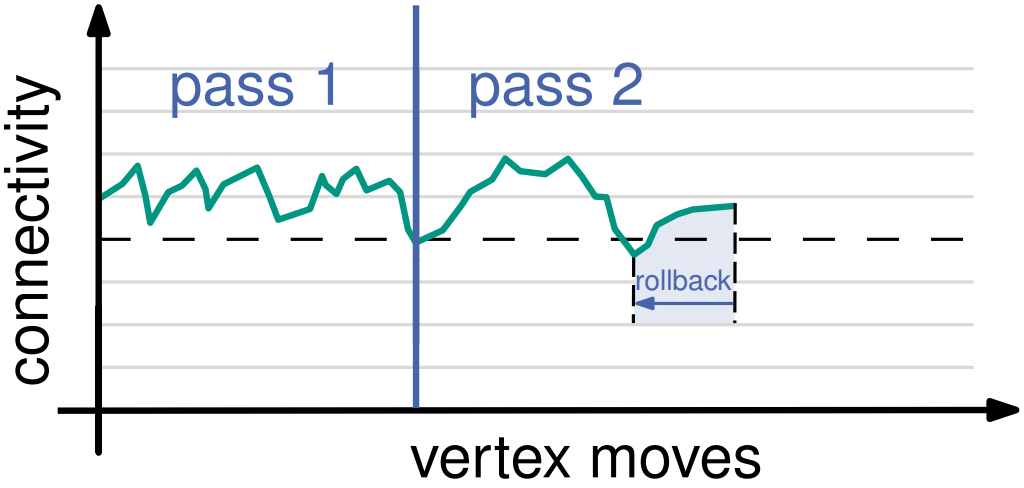
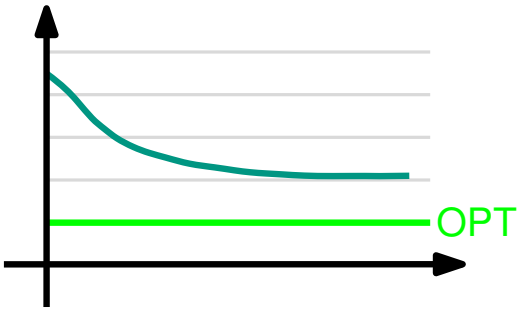
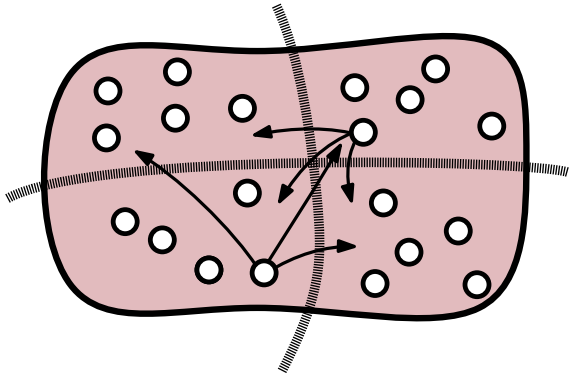


✗ get **stuck** in local optima



✗ large edges ⇝ **zero** gain moves



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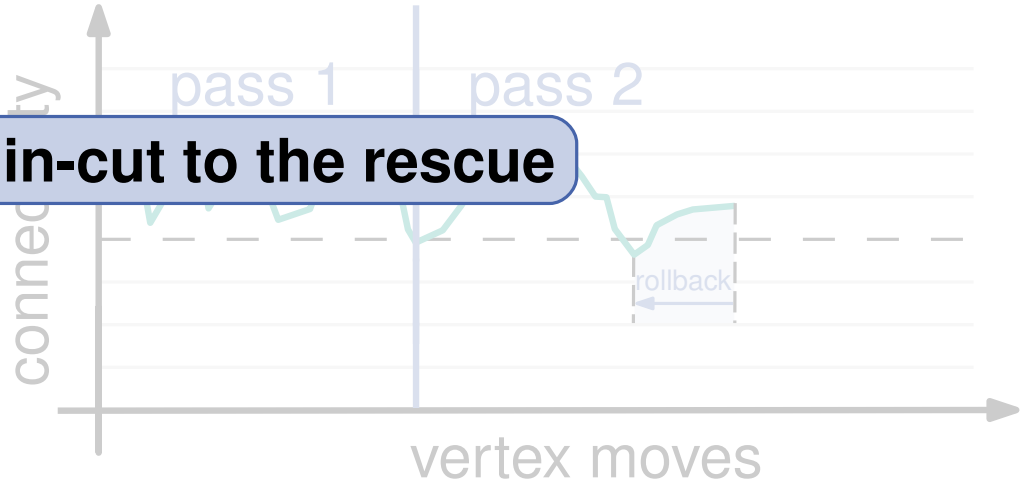
max-flow-min-cut to the rescue



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max-flow-min-cut to the rescue

Issues?

- only 2-way
- what are flows on hypergraphs?
- not balanced

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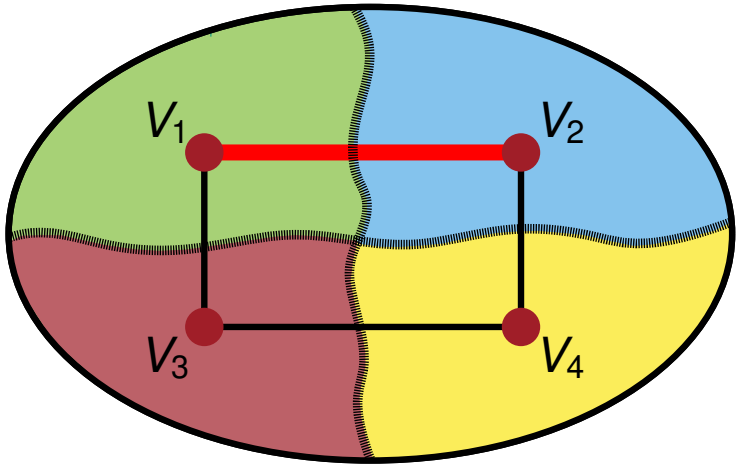
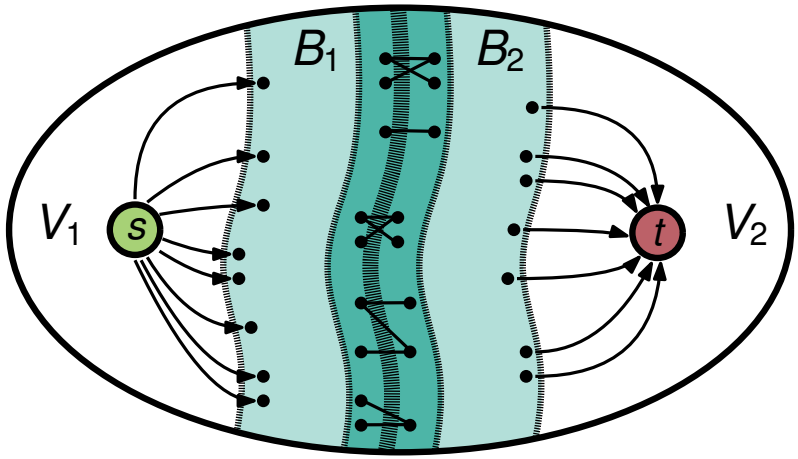
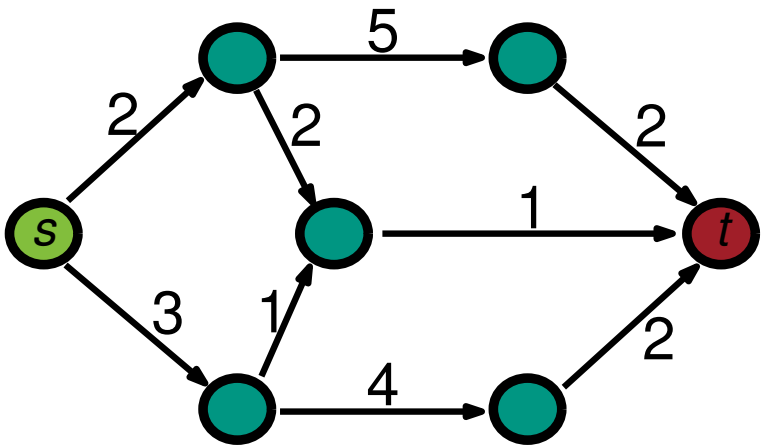
Flow-Based Refinement in KaHyPar



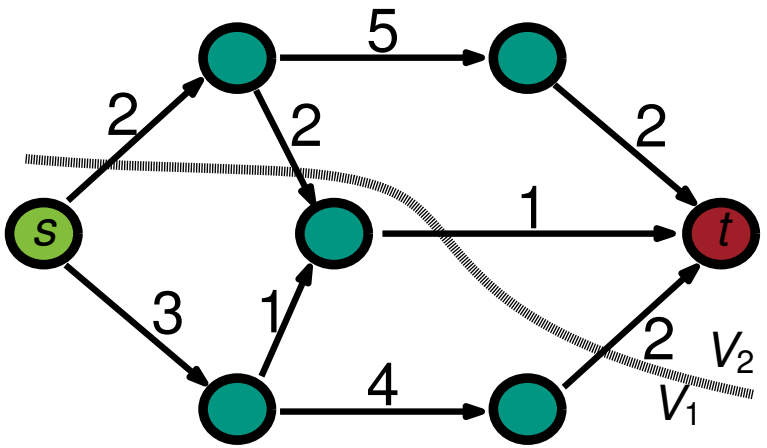
select two adjacent blocks for refinement



build graph-based flow model



solve flow problem



find more balanced minimum cut

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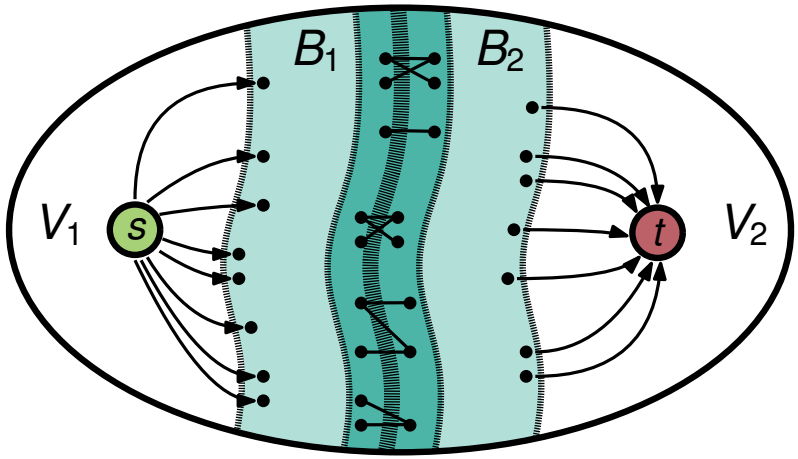
Flow-Based Refinement in KaHyPar



build graph-based flow model

- either: restrict flow model size so that balance is guaranteed
- or: make it a little larger, hope for balance. if not $\rightsquigarrow$ scale down again

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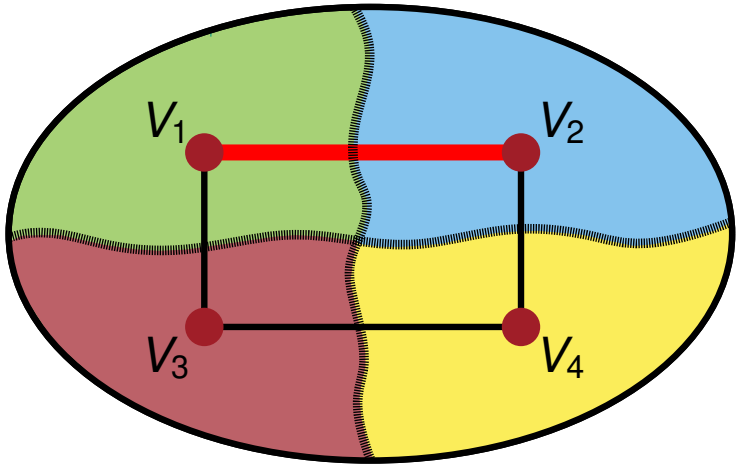

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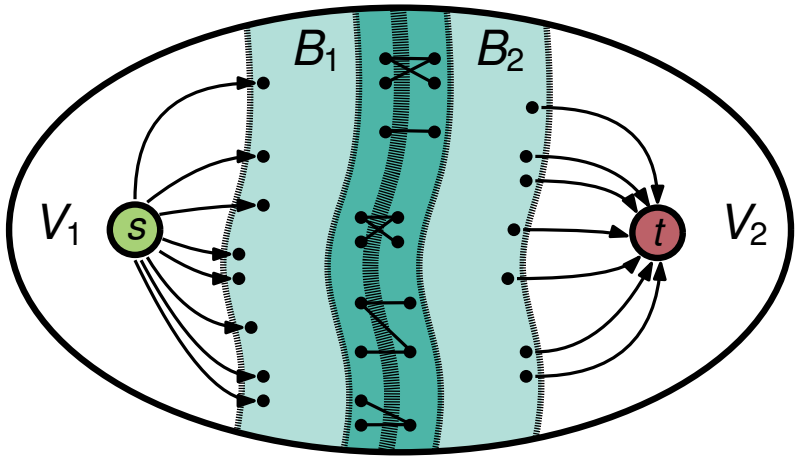

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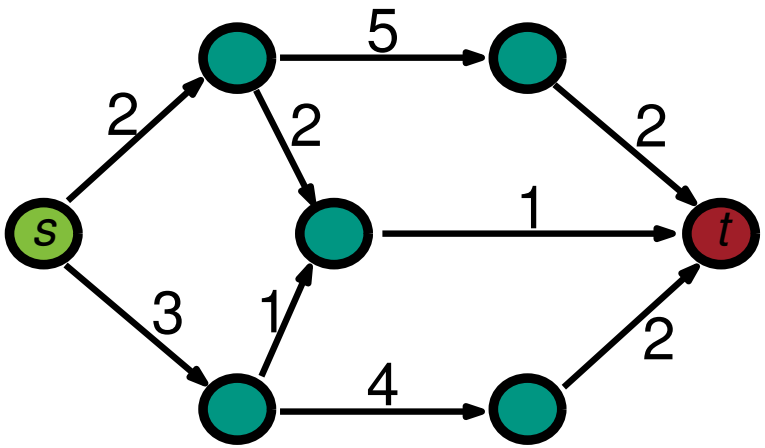

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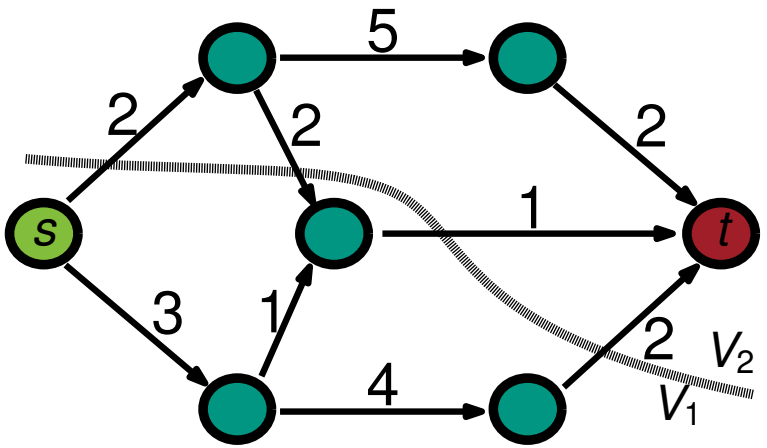

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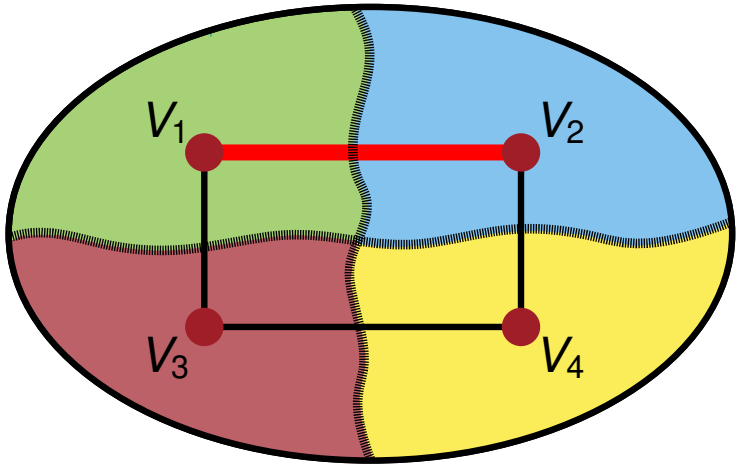

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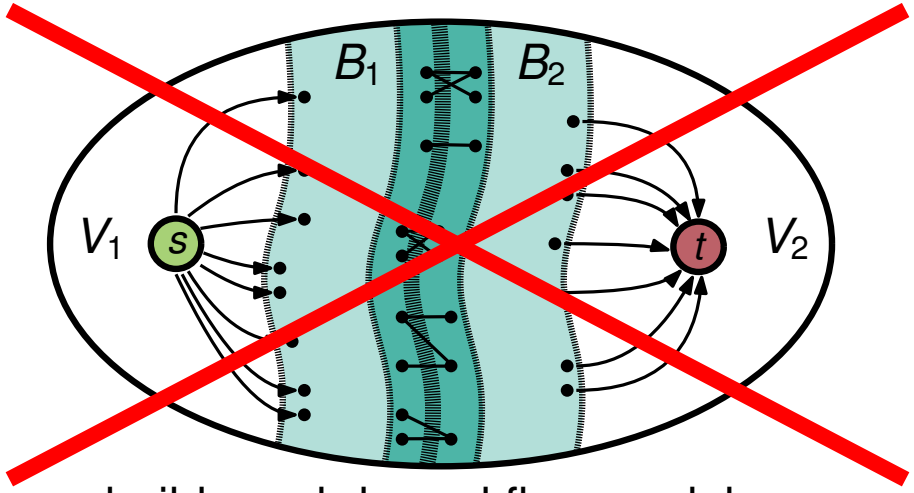

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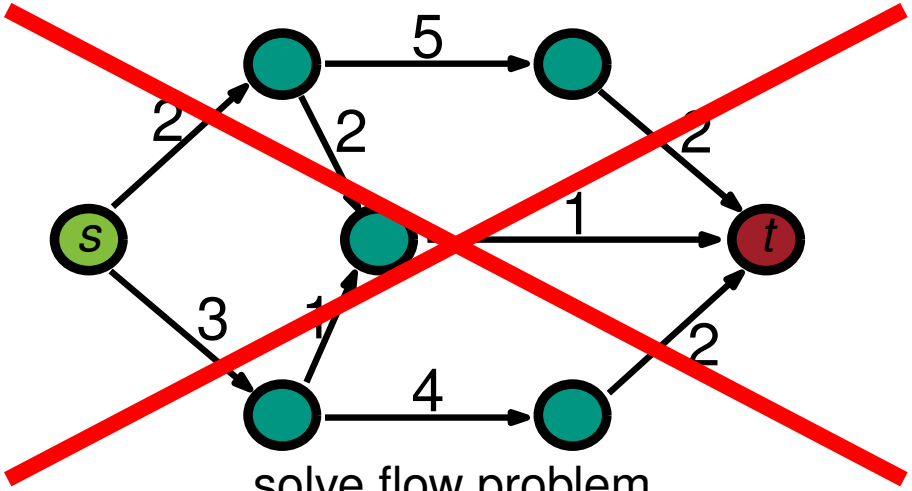

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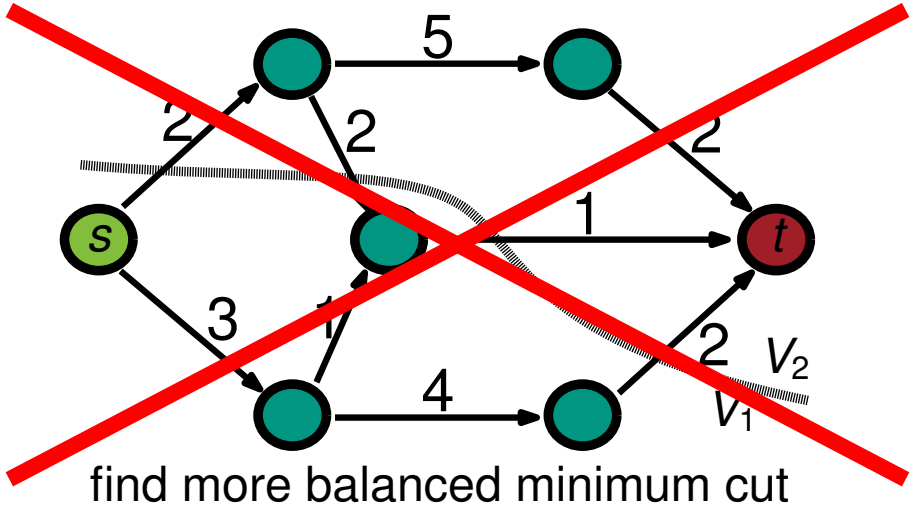

find more balanced minimum cut

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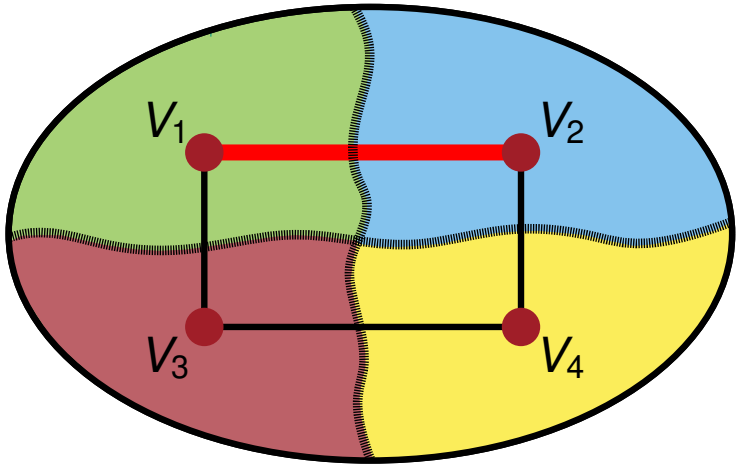
The new KaHyPar-HFC



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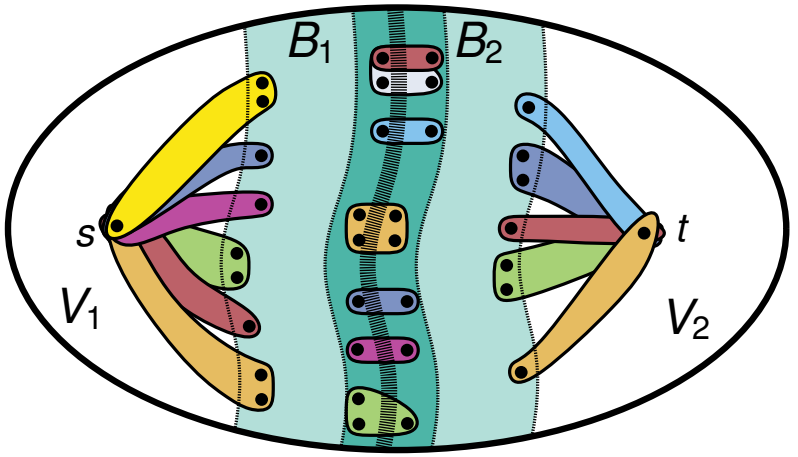
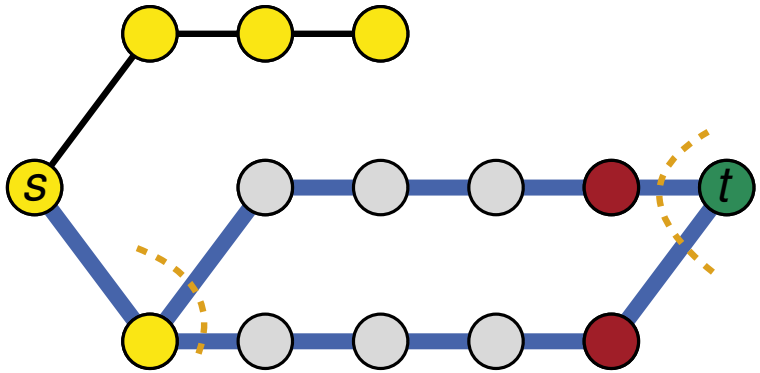


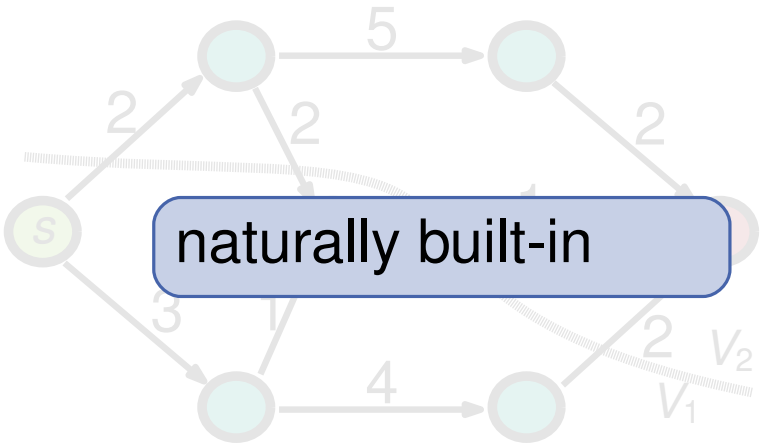
flows directly on hypergraph



use FlowCutter

[Hamann, Strasser JEA18]

[Gottesbüren, Hamann, Wagner ESA19]



find more balanced minimum cut

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The new KaHyPar-HFC

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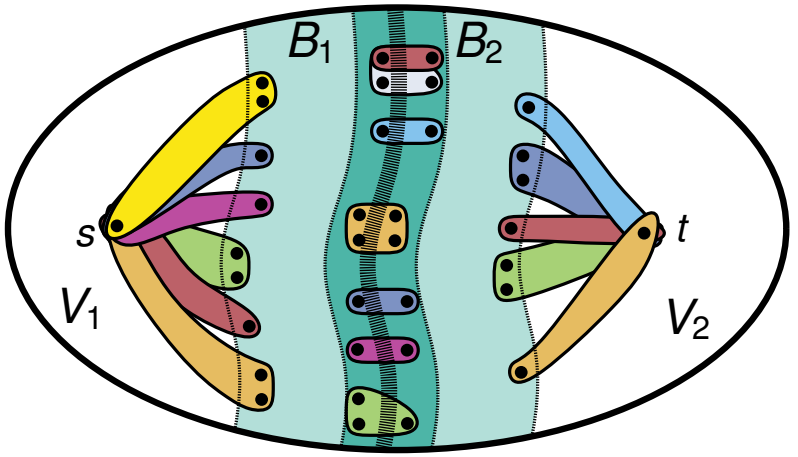

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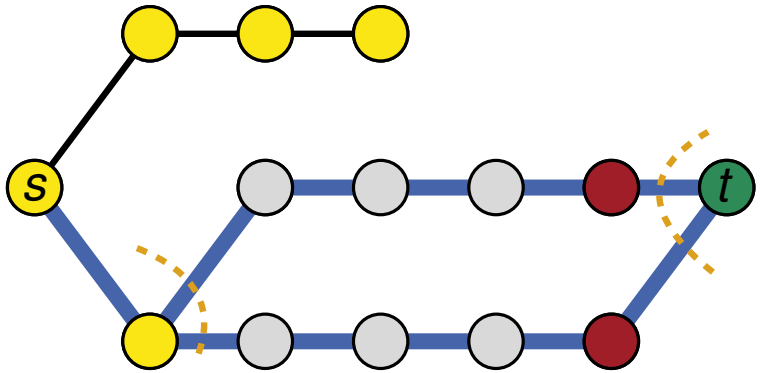

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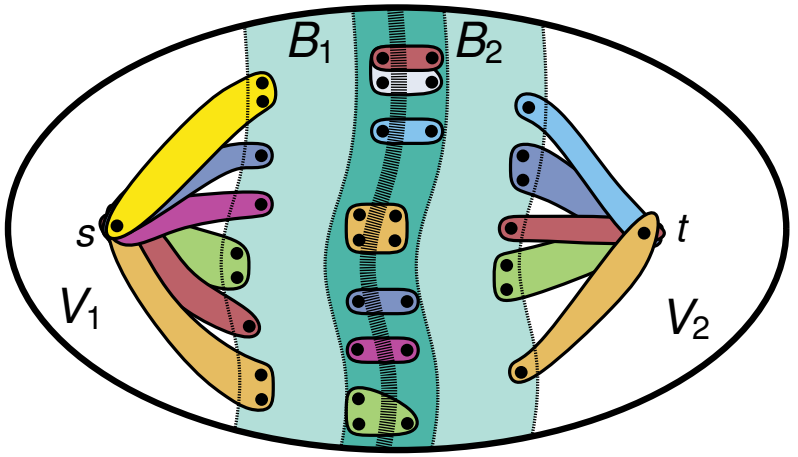

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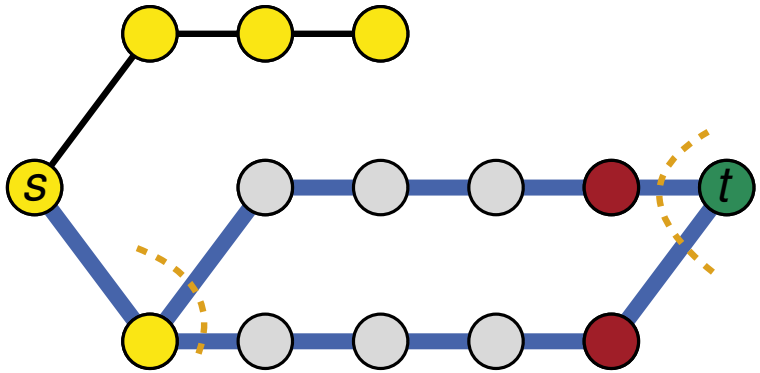

use FlowCutter
[Hamann, Strasser JEA18]
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what's new for FlowCutter?

- weighted instances
- new guidance

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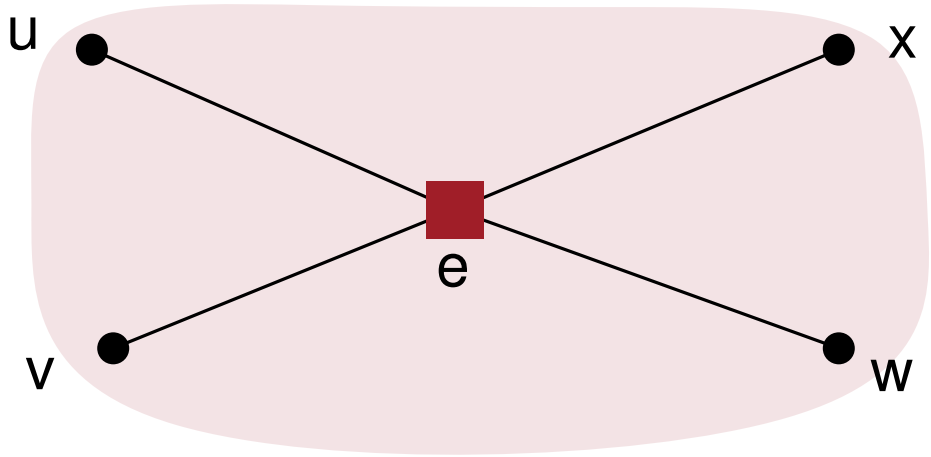
Flows on Hypergraphs



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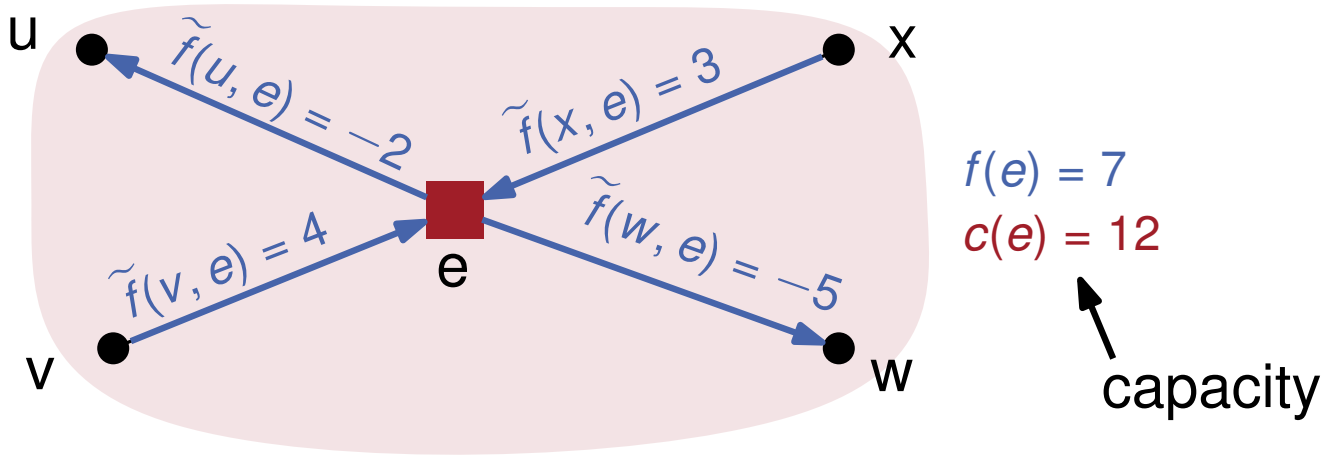
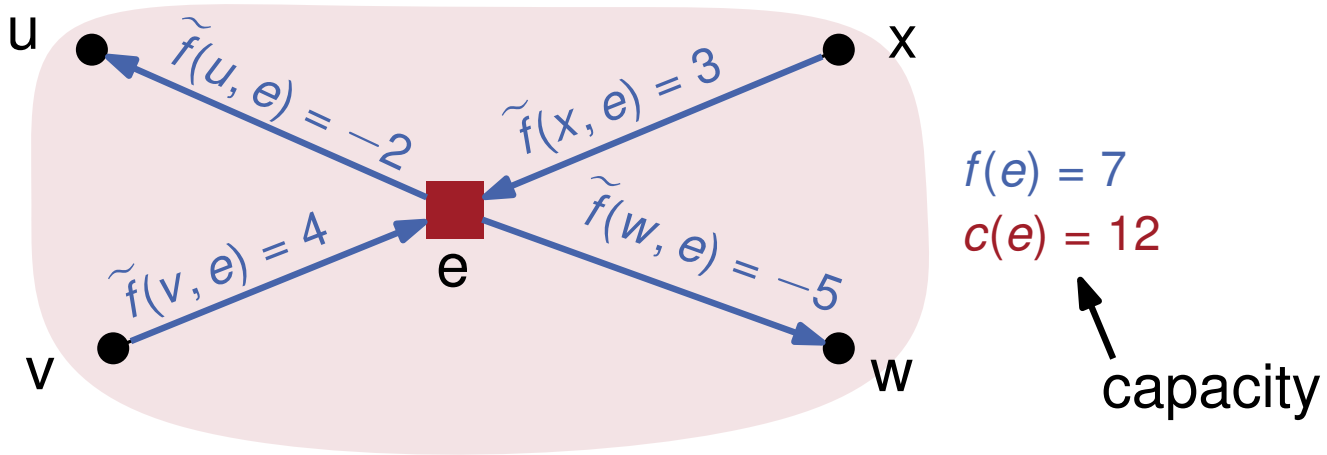


- $\tilde{f}(u, e) > 0 \Rightarrow u$ sends flow into e
- $\tilde{f}(u, e) < 0 \Rightarrow u$ receives flow from e

Flows on Hypergraphs



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  $\max(0, -\widetilde{f}(u,e))$ $\max(0, \widetilde{f}(v,e))$

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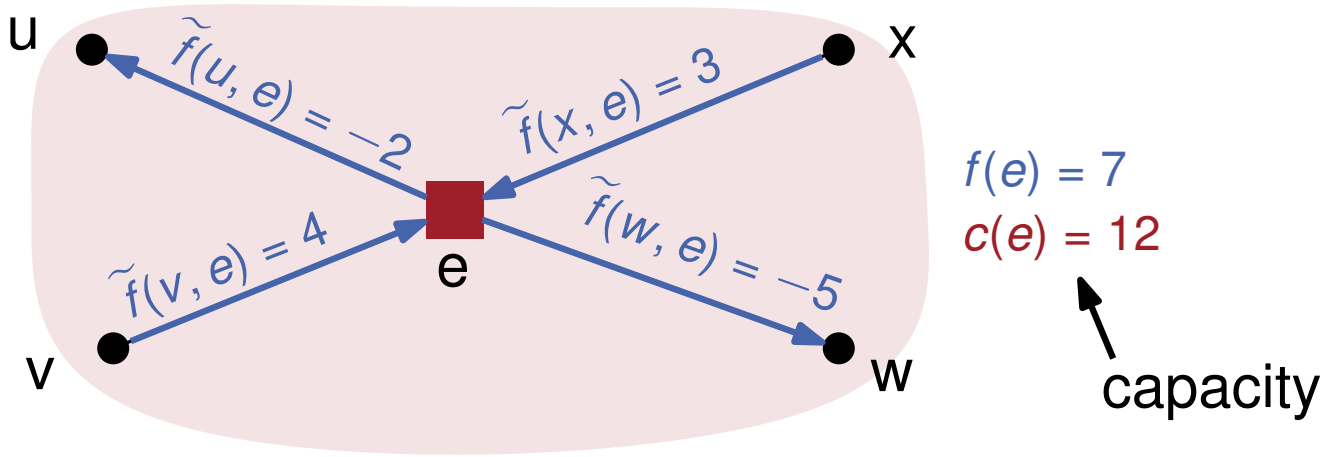
Flows on Hypergraphs



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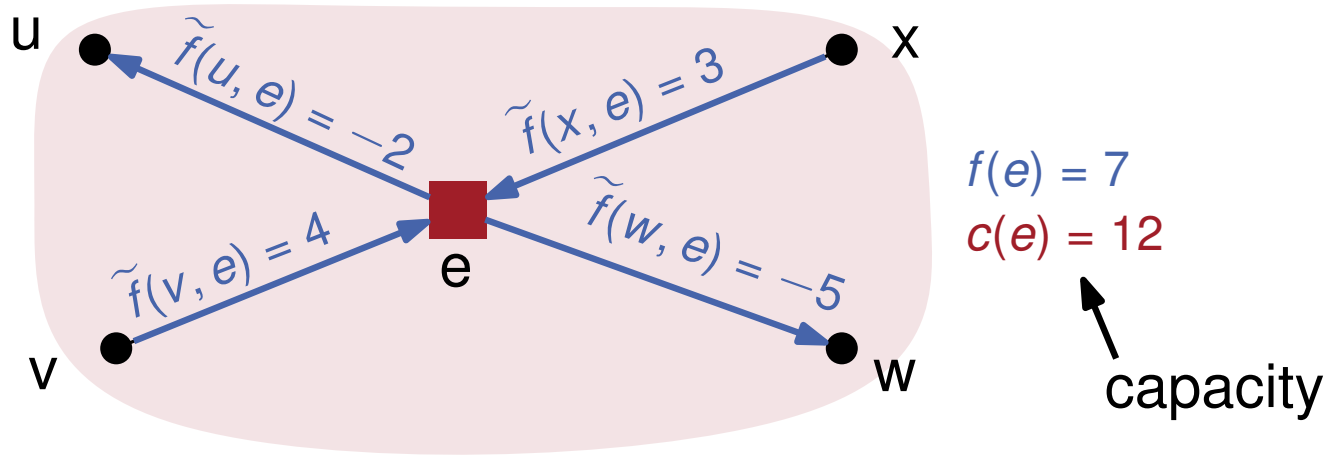

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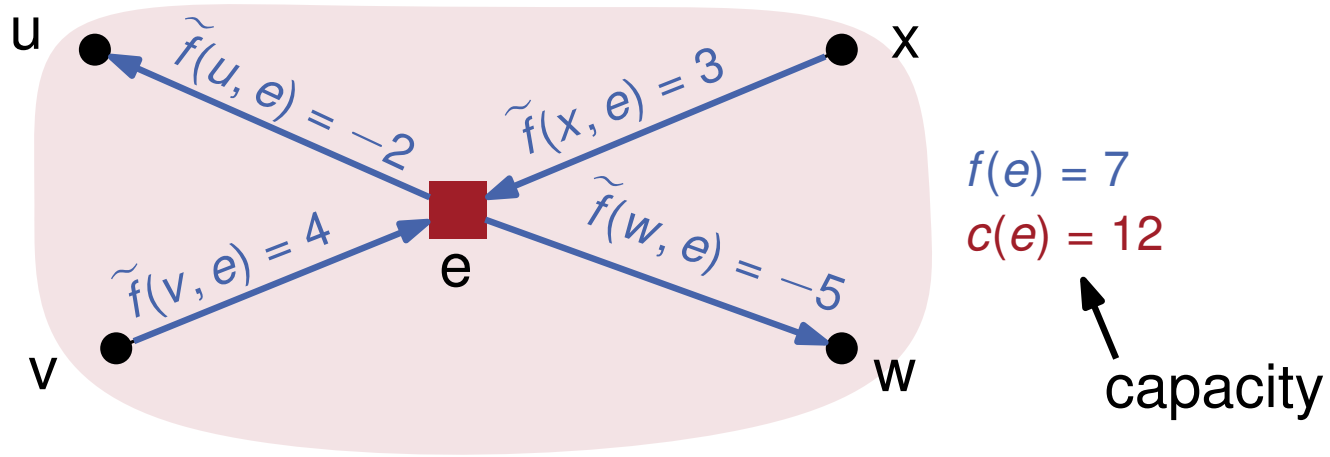

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- $\text{rcap}(e,u,v) = c(e) - f(e) + \tilde{f}(u,e)^- + \tilde{f}(v,e)^+ = (12-7) + 2 + 4 = 11$

  $\max(0, -\tilde{f}(u,e))$ $\max(0, \tilde{f}(v,e))$

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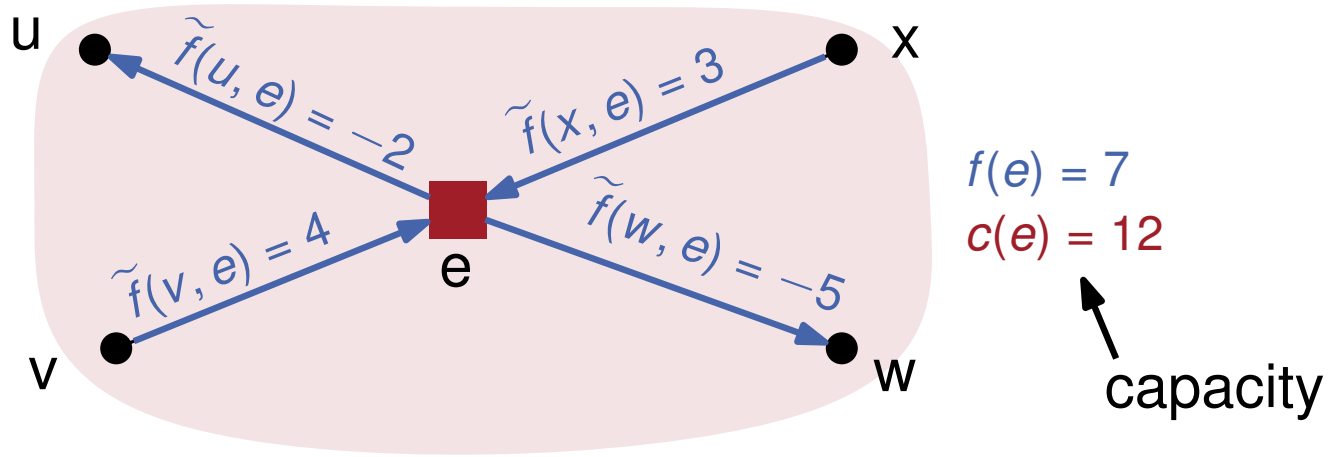
Flows on Hypergraphs



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  $\max(0, -\tilde{f}(u,e))$ $\quad$ $\max(0, \tilde{f}(v,e))$
- can implement *any* flow algorithm by treating nets like vertices
- we use Dinic

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Dinic

- residual BFS to compute distance labels
- residual DFS to find edge-disjoint shortest augmenting paths
- repeat until no flow augmented

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Dinic

- residual BFS to compute distance labels
- residual DFS to find edge-disjoint shortest augmenting paths
- repeat until no flow augmented

Distance labels for hypergraphs

- $d[u]$ for vertices
- $d_i[e]$ for pushing flow to flow-sending pins $\widetilde{f}(v, e) > 0$
- $d_o[e]$ for pushing flow to all pins. set if $c(e) - f(e) + \widetilde{f}(u, e)^- > 0$

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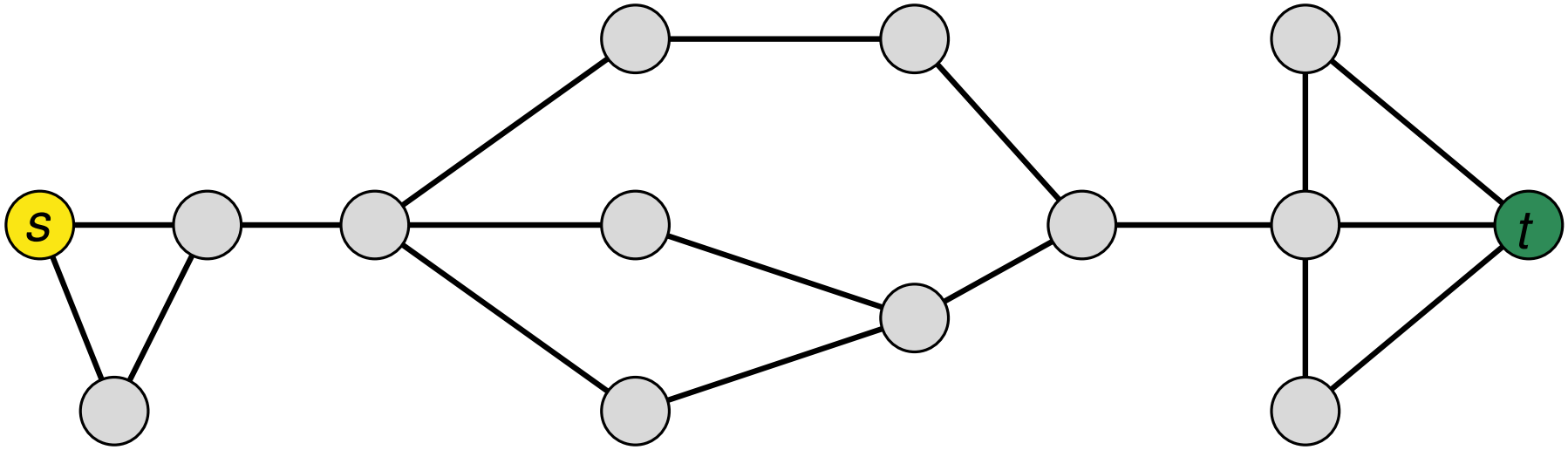
Dinic

- residual BFS to compute distance labels
- residual DFS to find edge-disjoint shortest augmenting paths
- repeat until no flow augmented

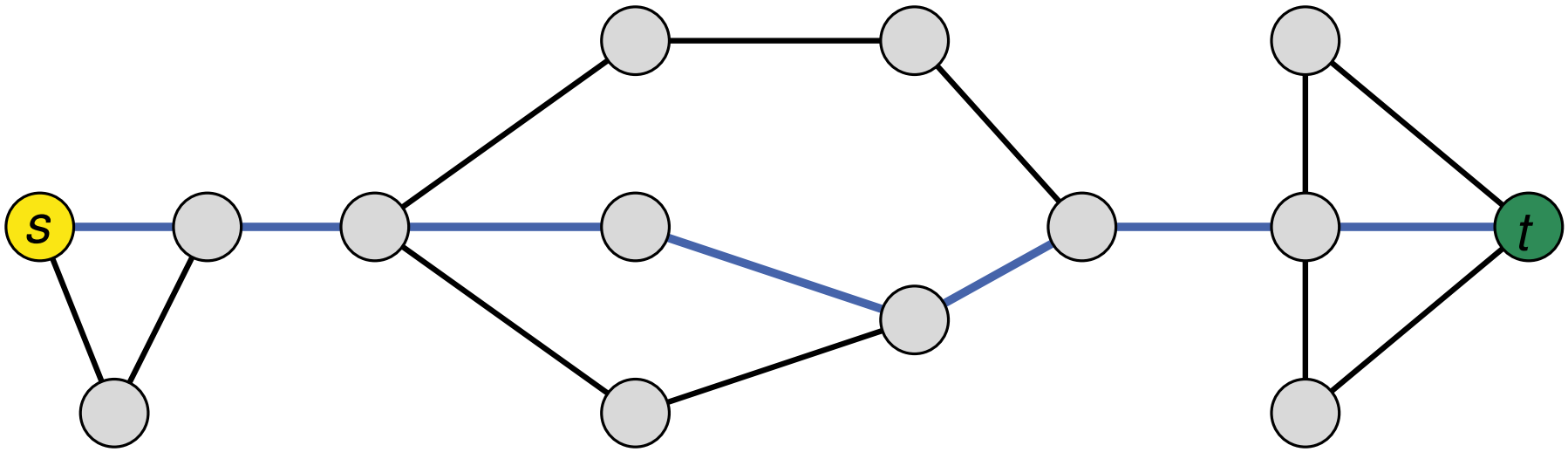
Optimizations

- capacity scaling
  - require $\geq \alpha$ residual capacity
  - if no flow augmented try with $\alpha \leftarrow \alpha/2$
- iterative DFS with stored iterators
- no allocations
- range of active values trick $\rightsquigarrow$ fast resets

FlowCutter

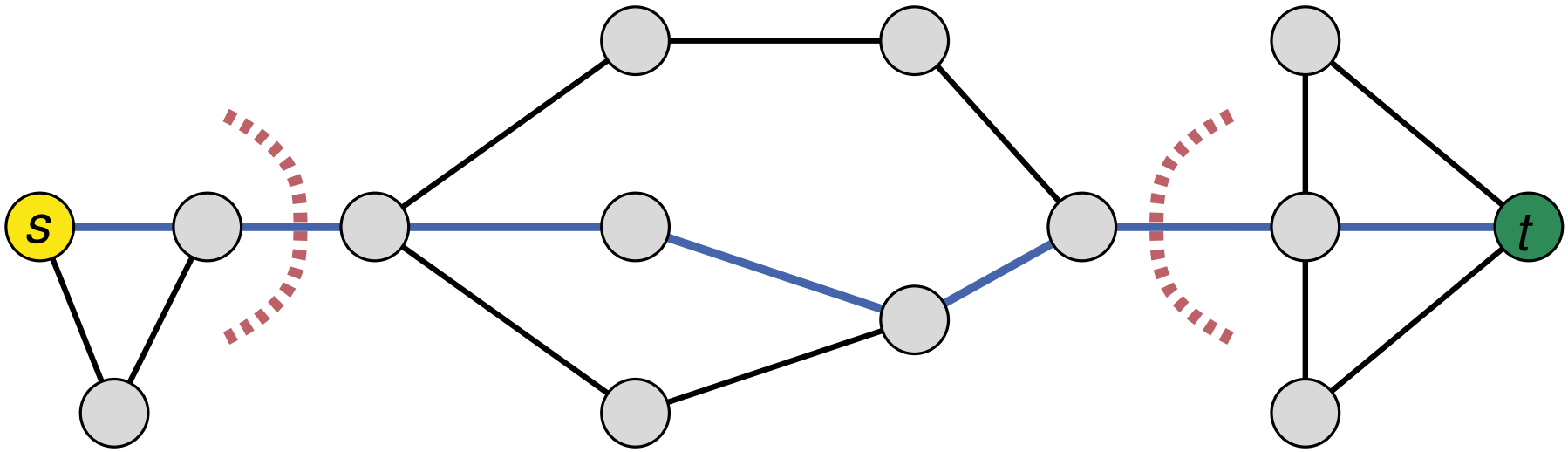


FlowCutter



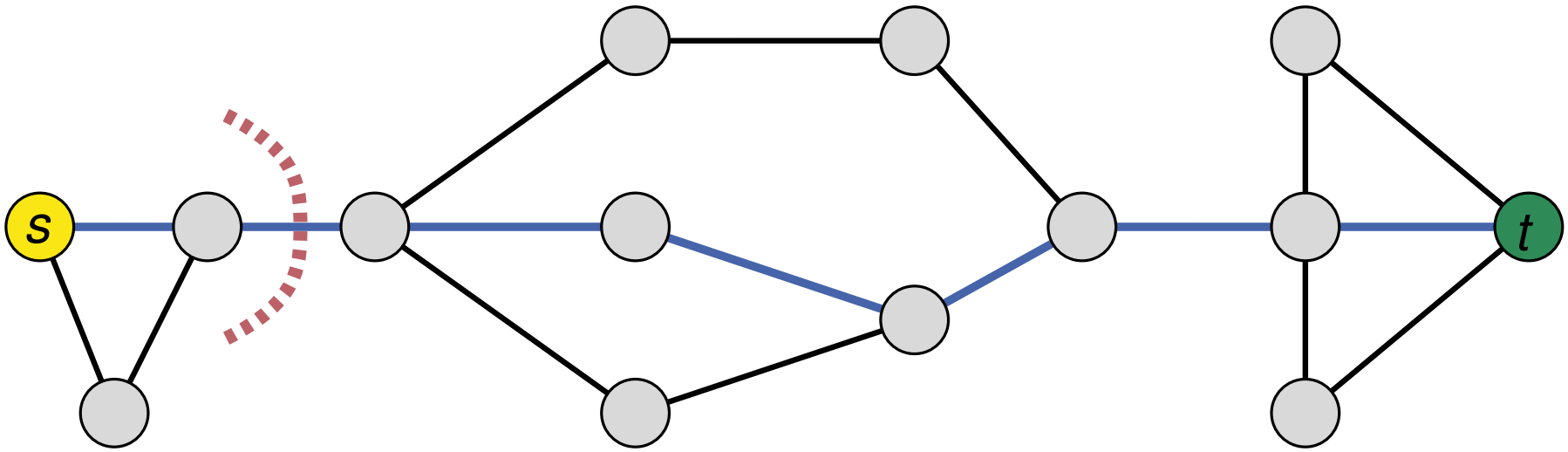
1. Augment flow

FlowCutter



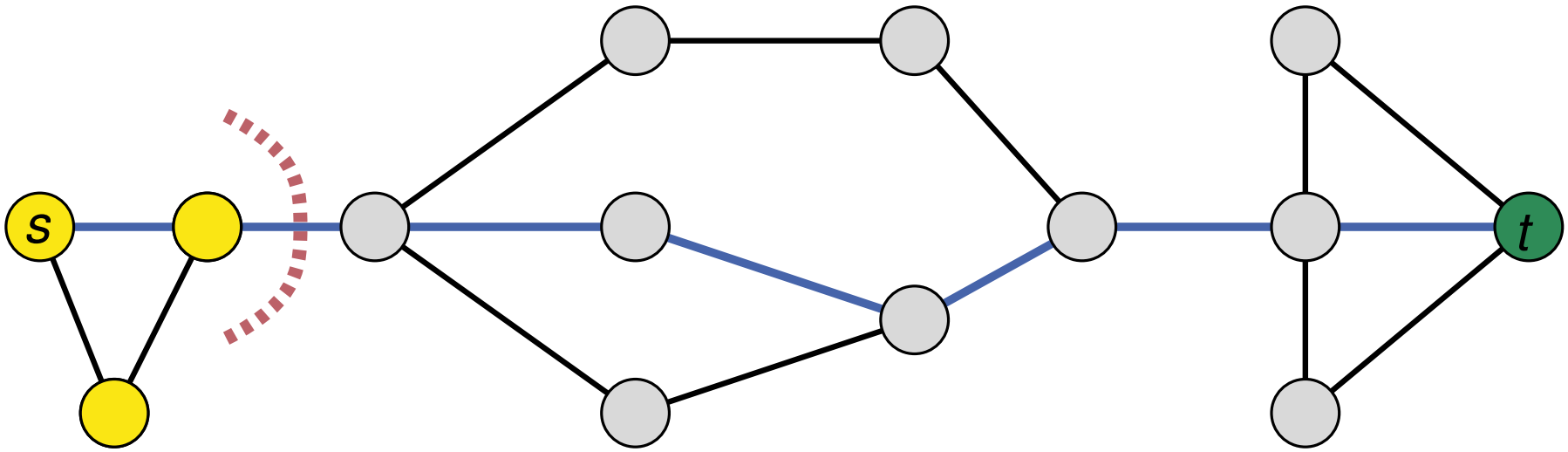
1. Augment flow
2. Find min *s*- and *t*-cut

FlowCutter



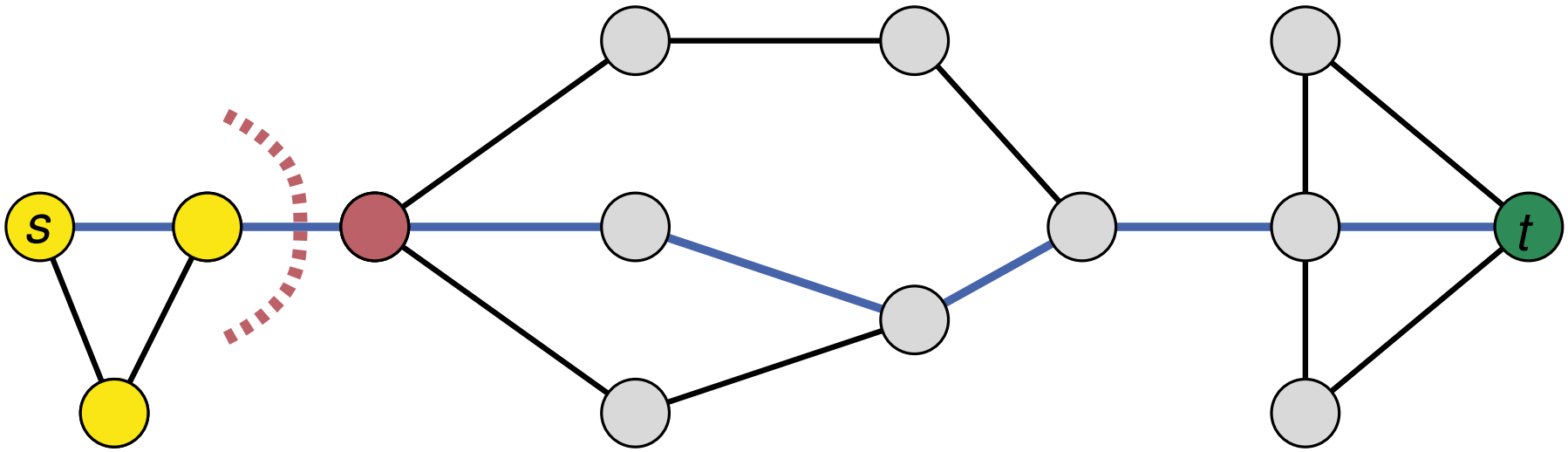
1. Augment flow
2. Find min *s*- and *t*-cut
3. Pick smaller side

FlowCutter



1. Augment flow
2. Find min *s*- and *t*-cut
3. Pick smaller side
4. Assimilate side

FlowCutter



1. Augment flow
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5. Pierce cut

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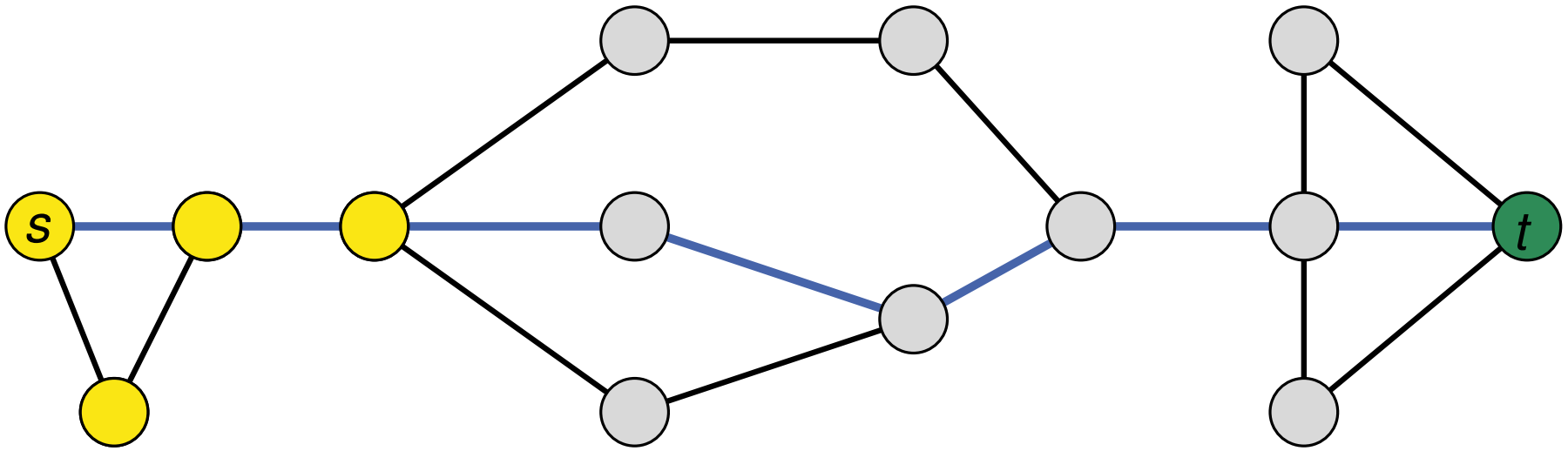
FlowCutter



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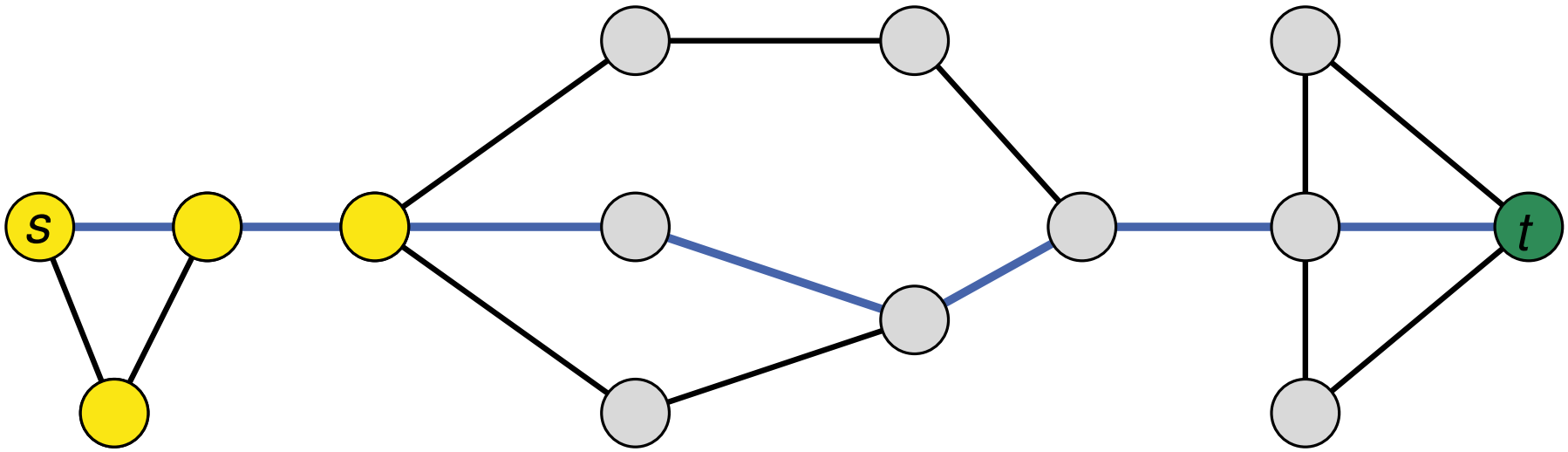
FlowCutter



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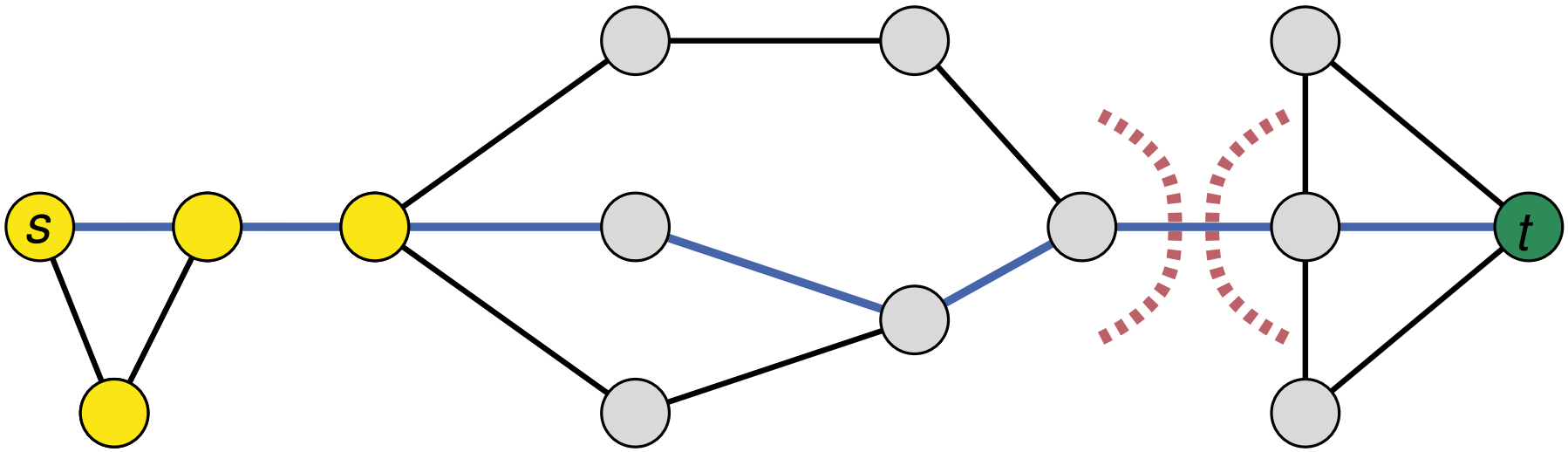

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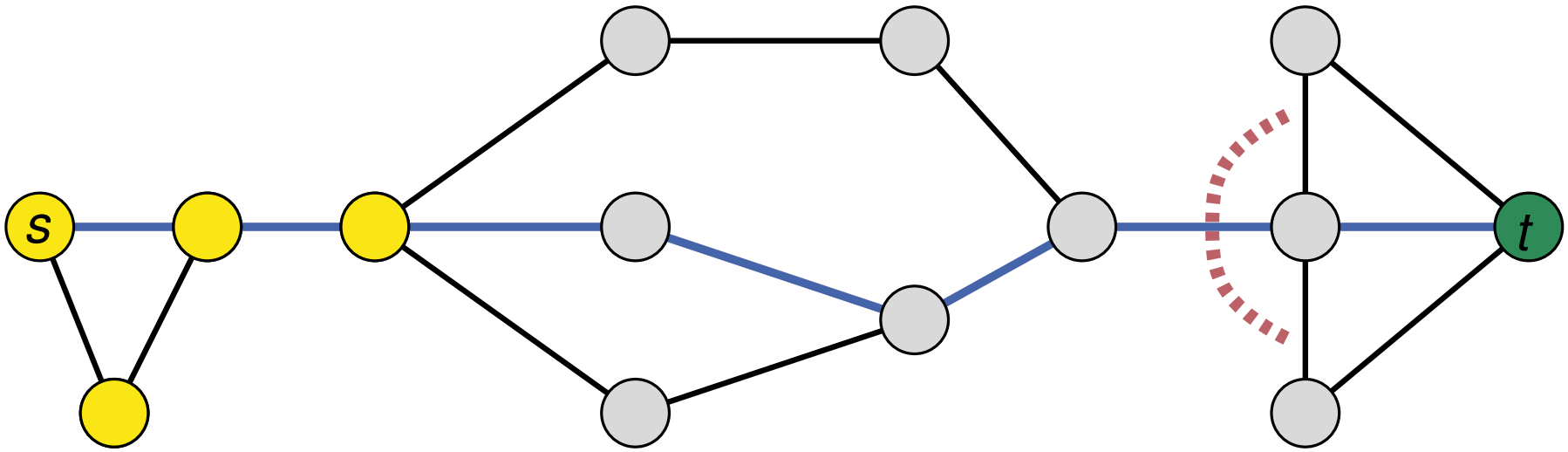

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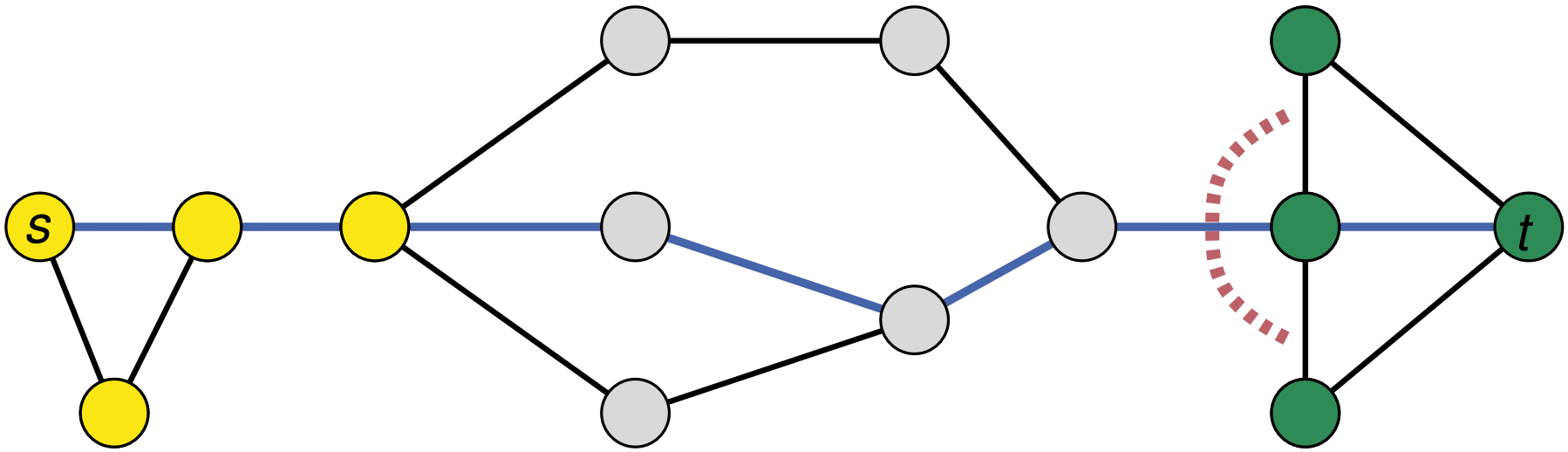

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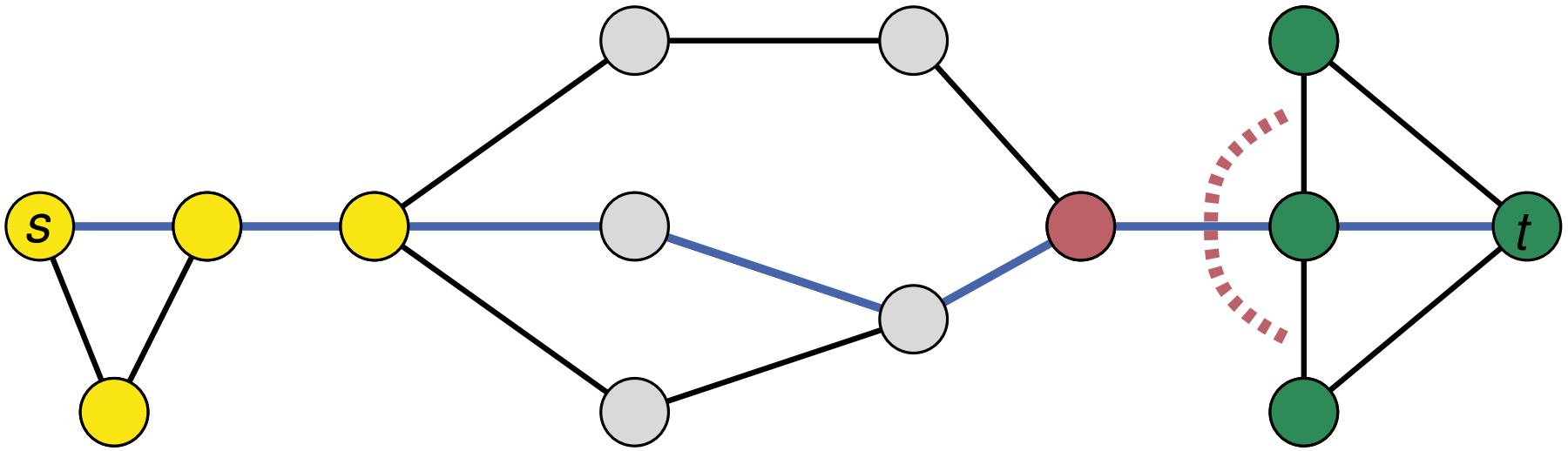

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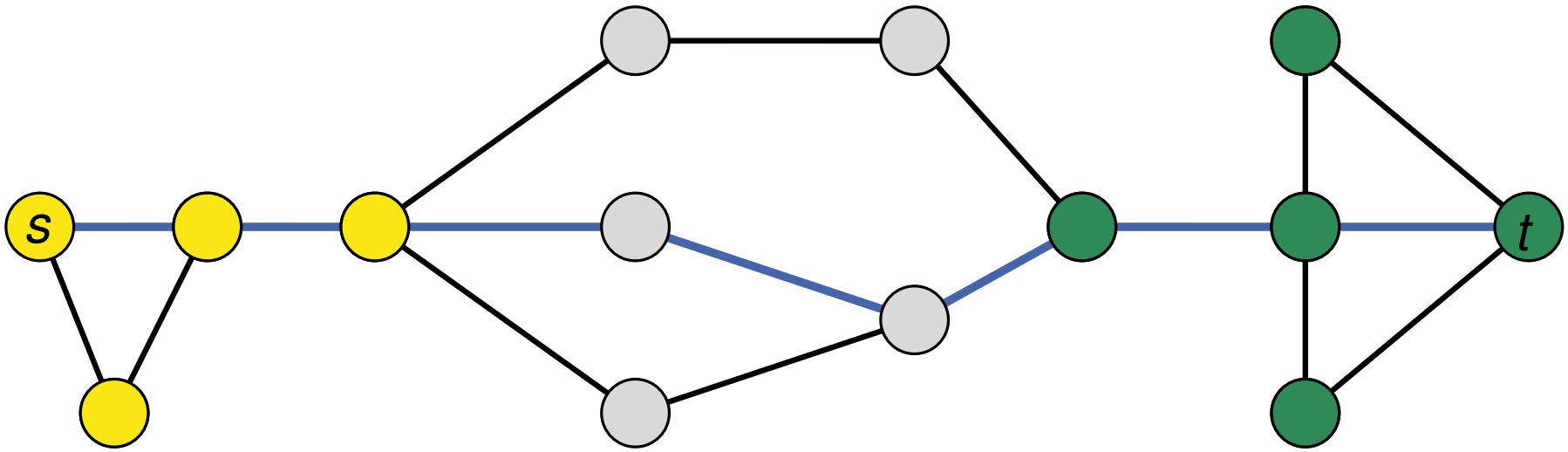

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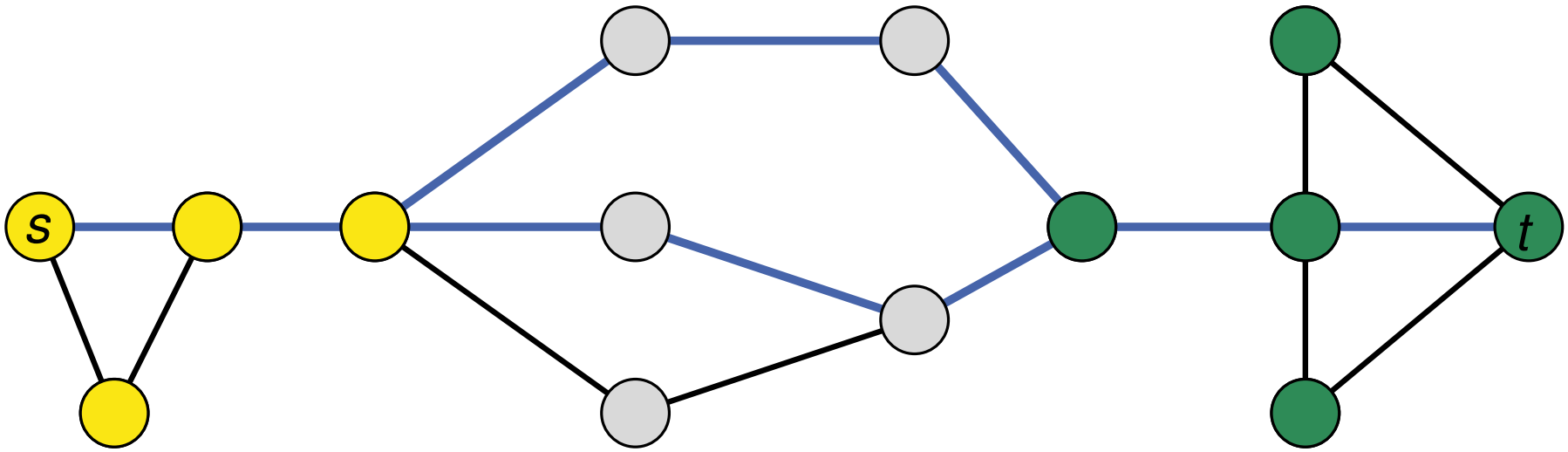

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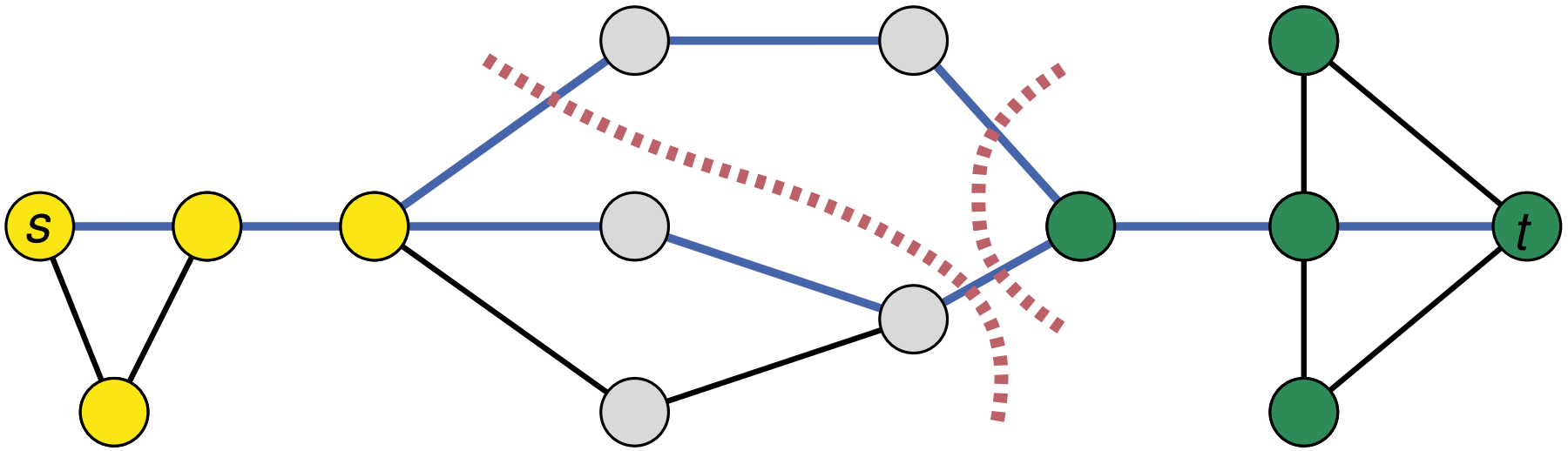

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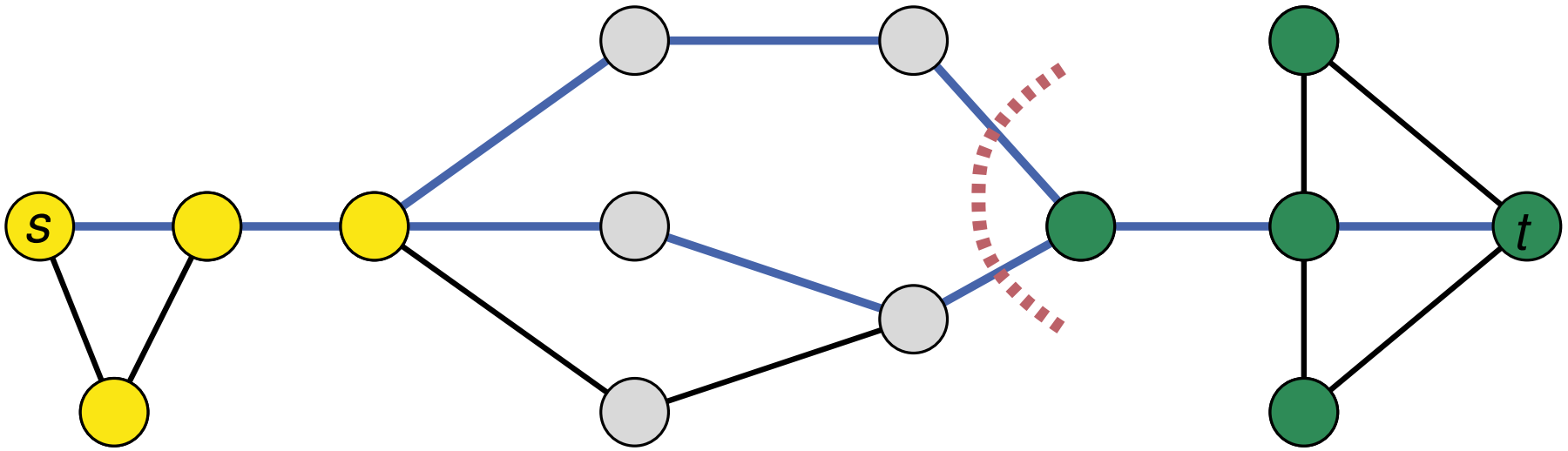

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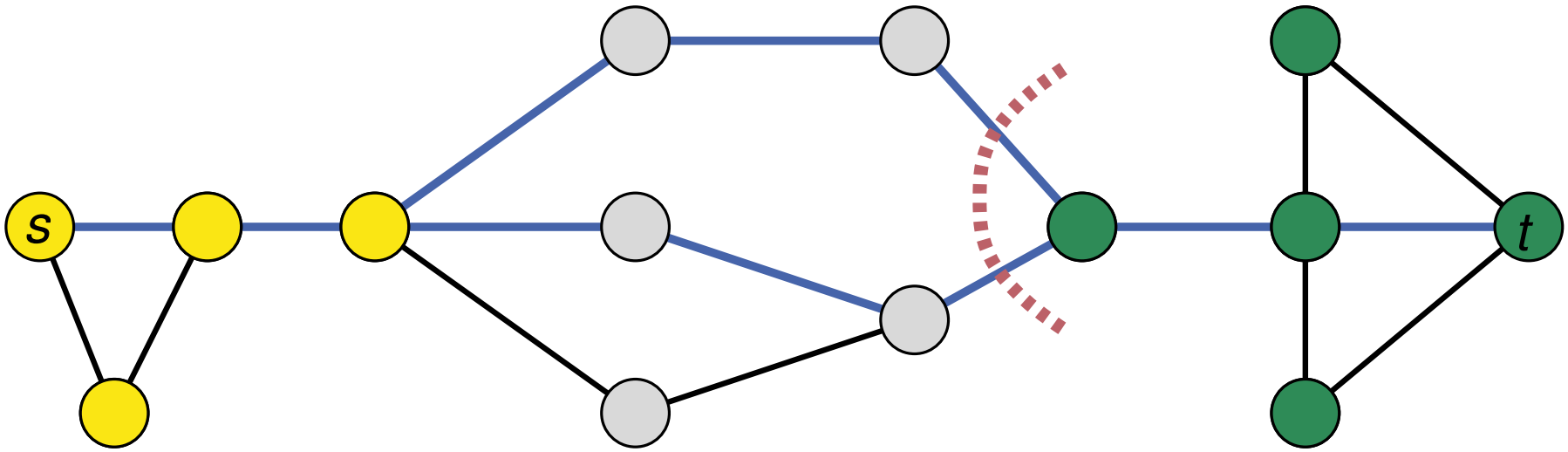

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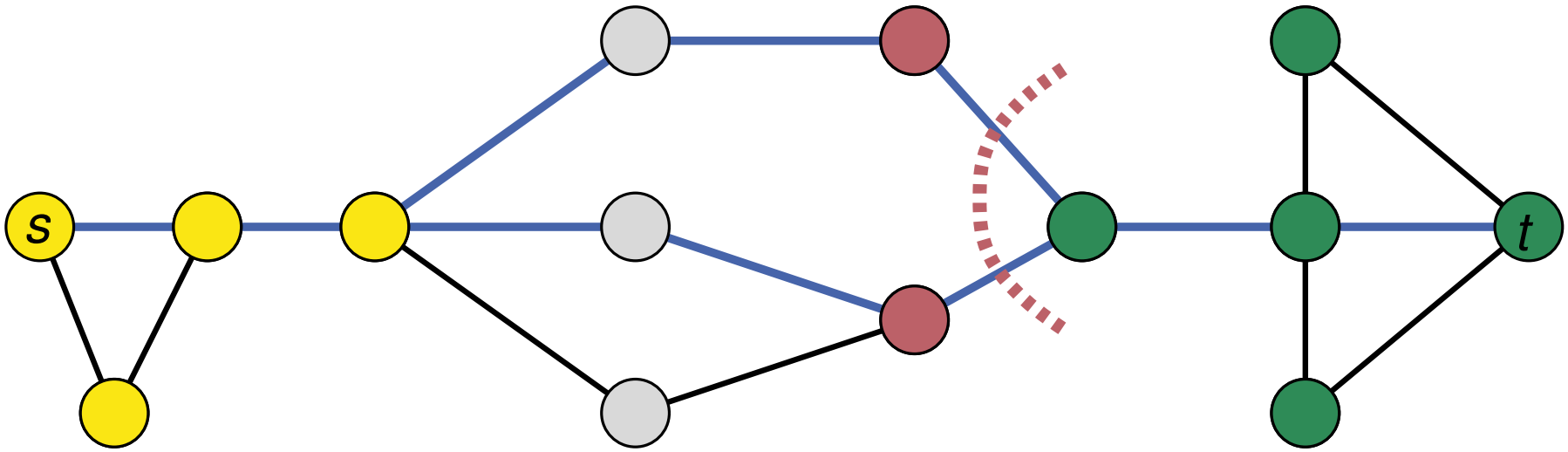

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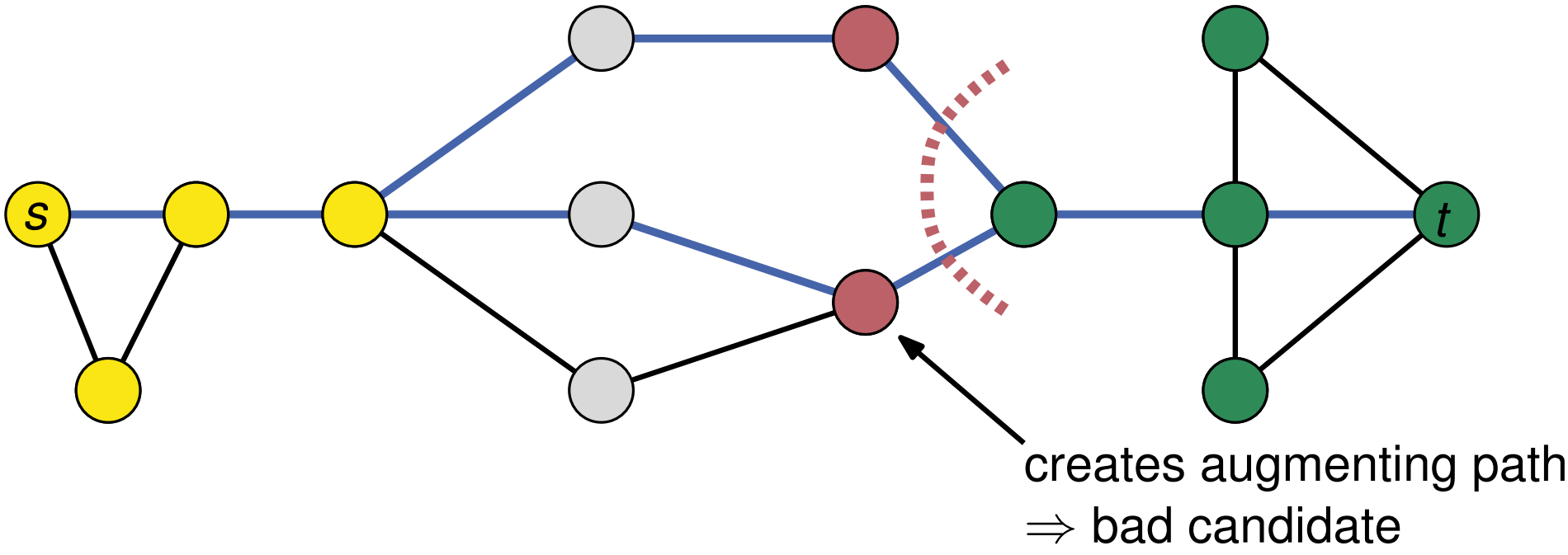

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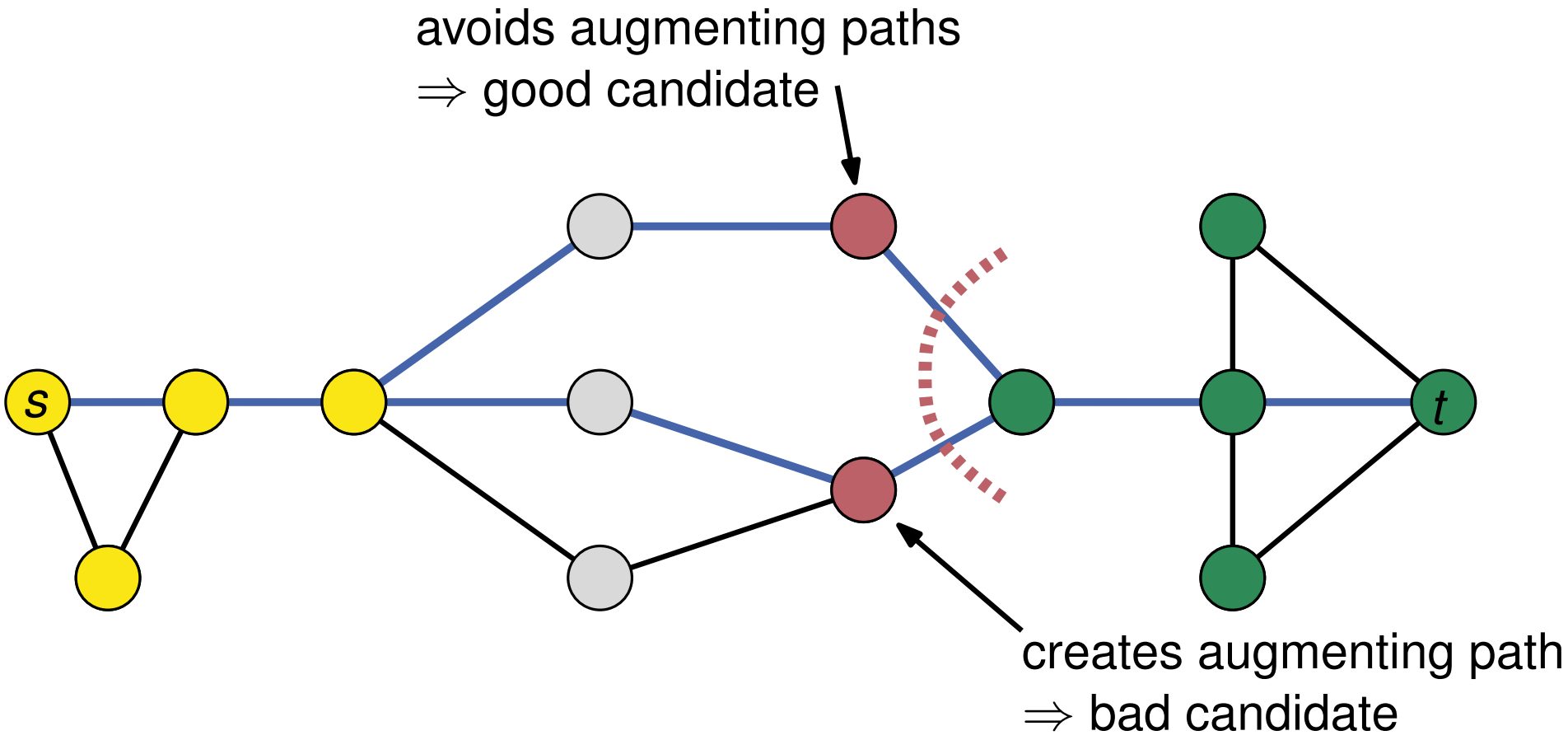

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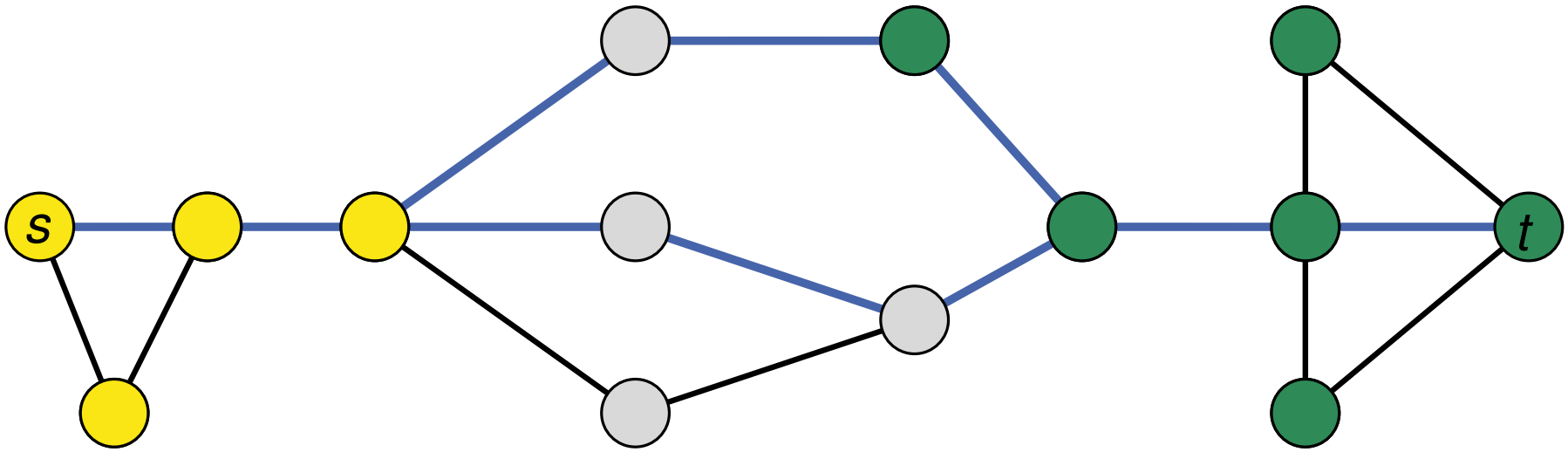

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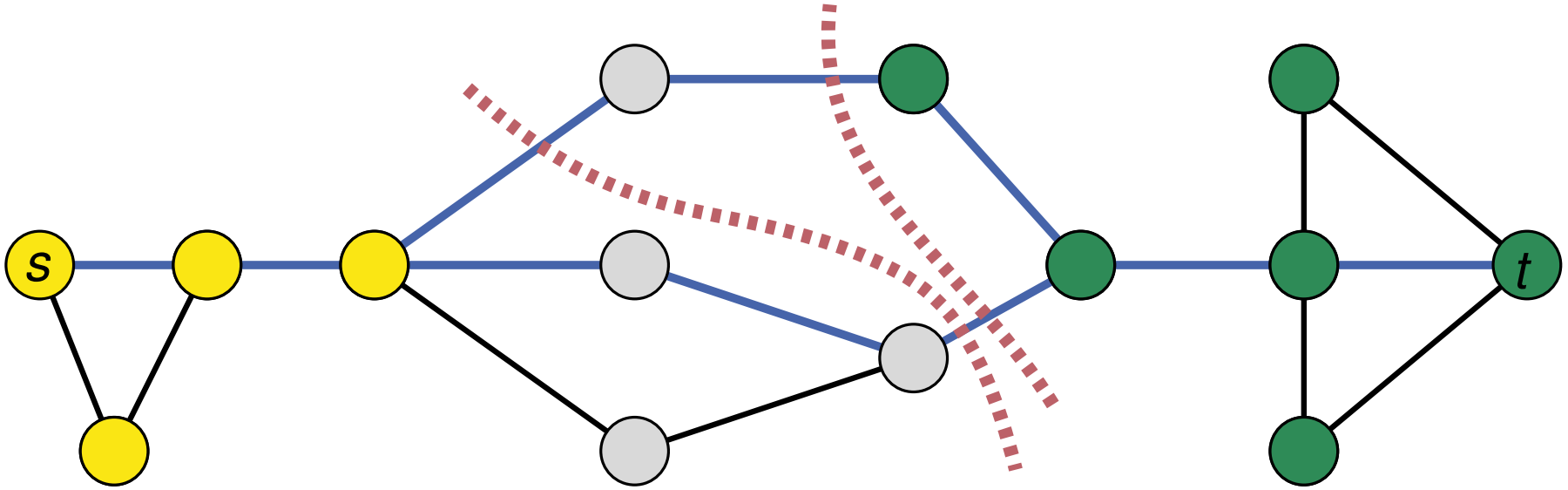

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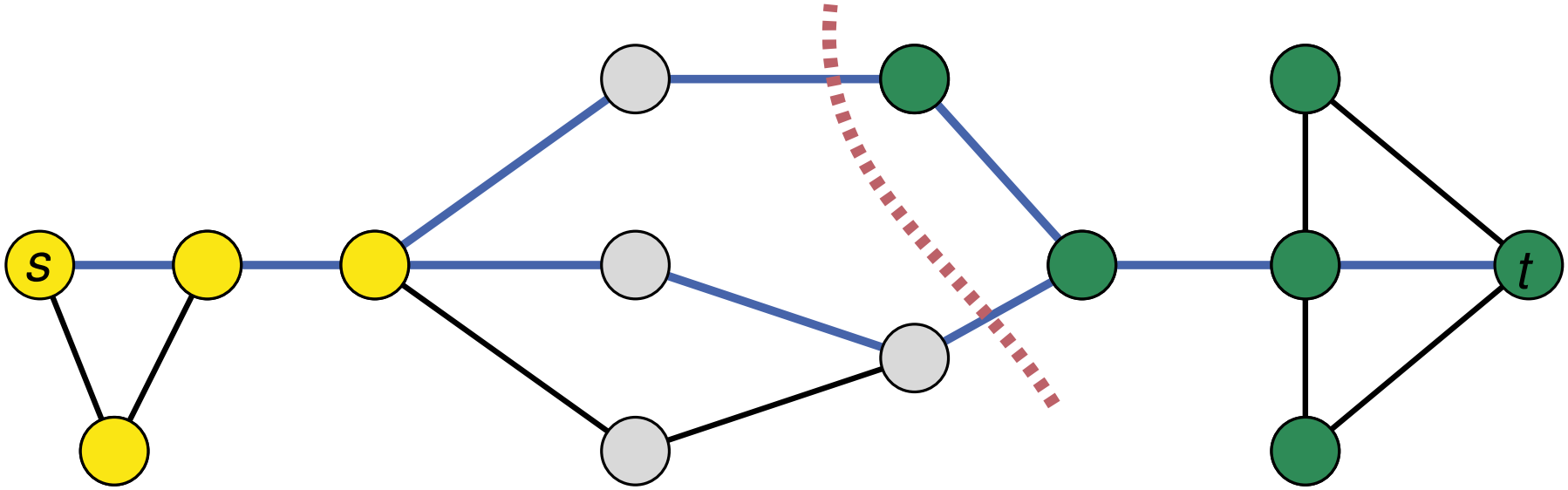

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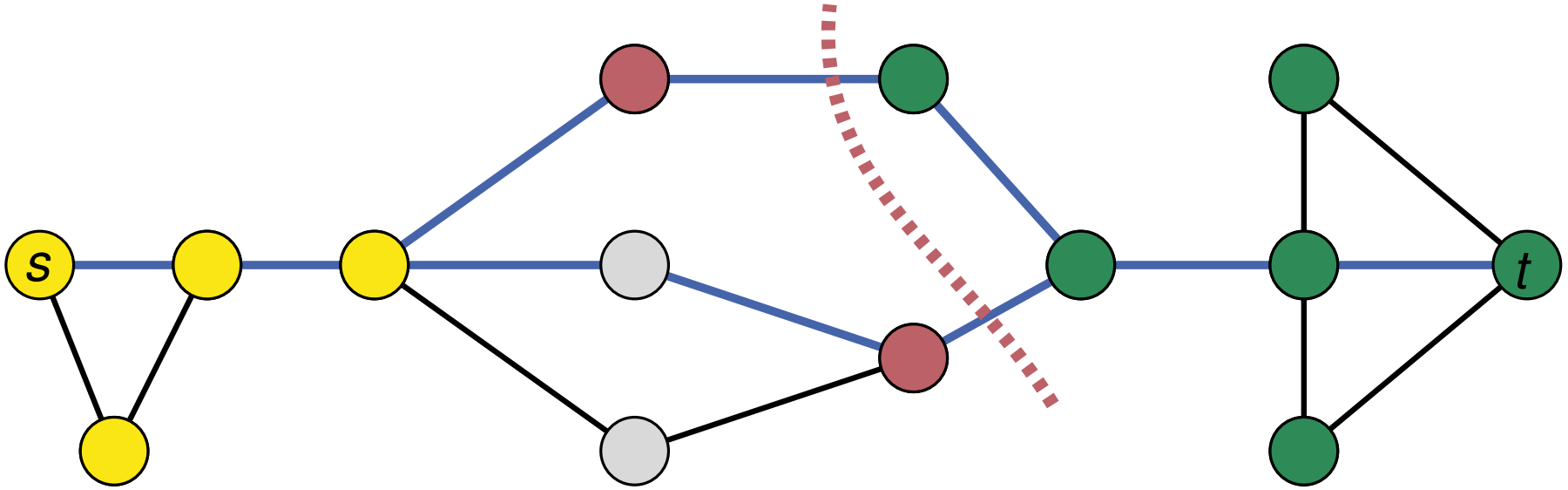

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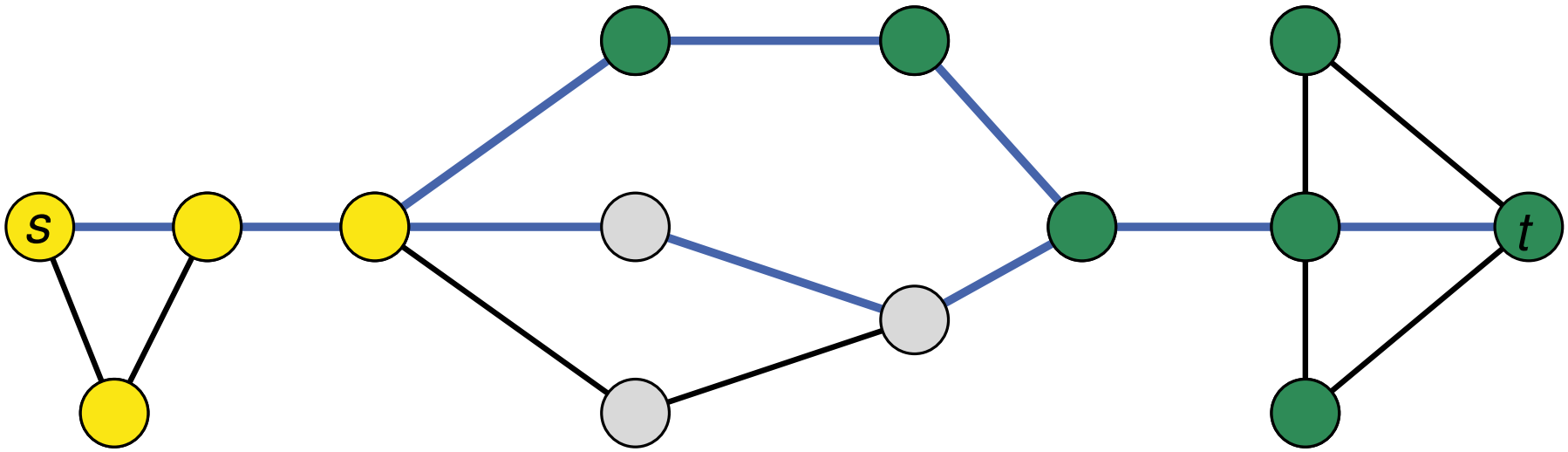

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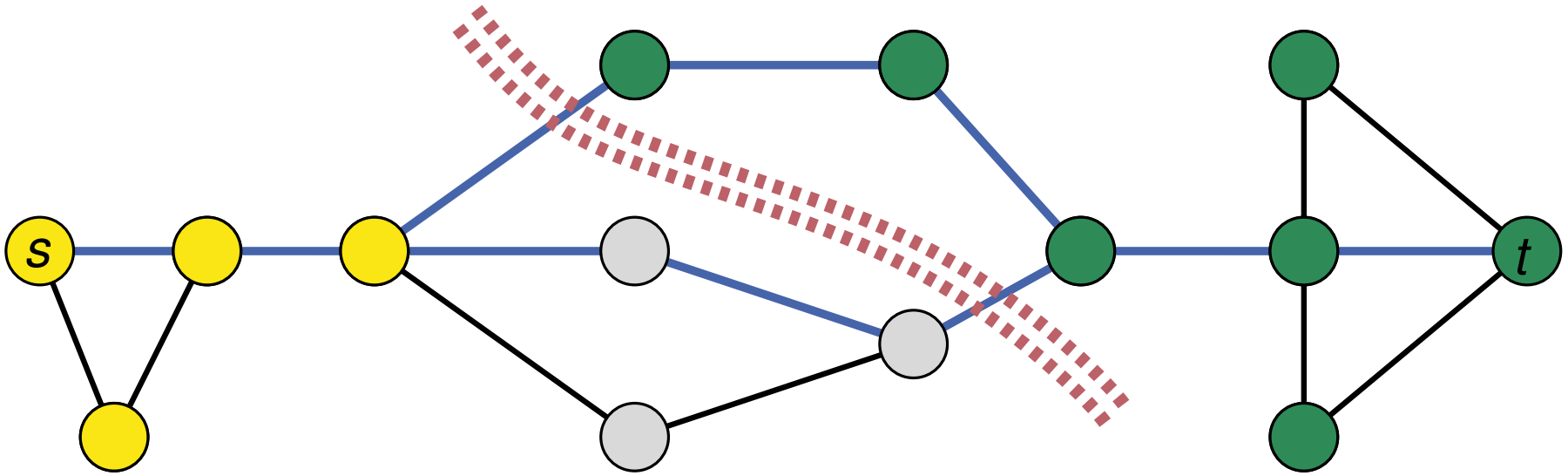

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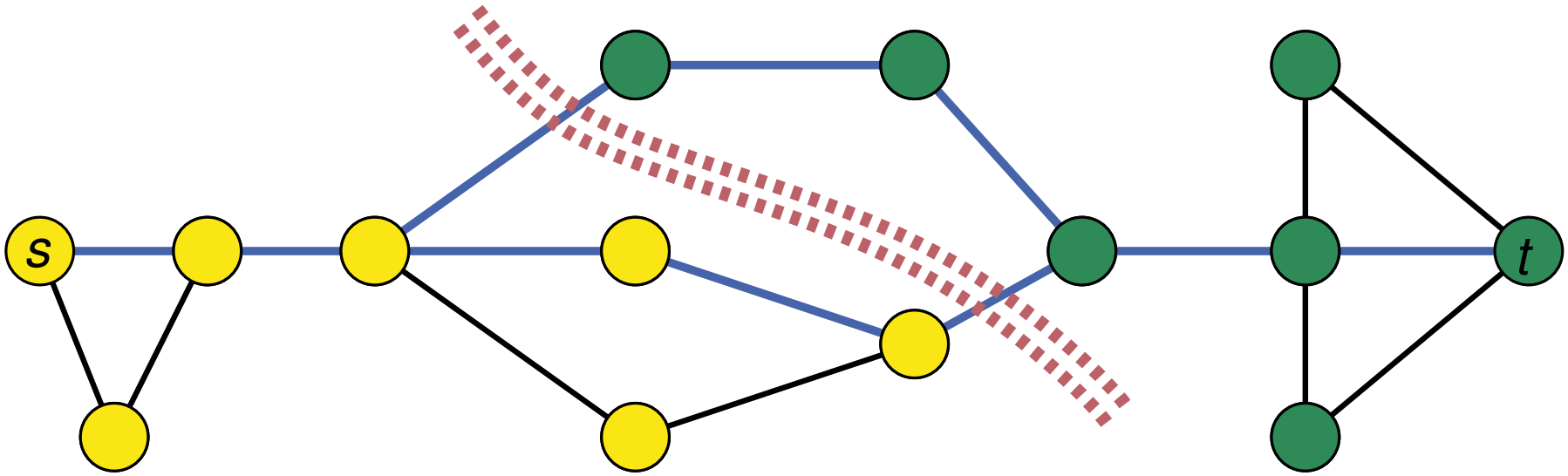

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Experimental Setup

- Intel Xeon E5-2670 @ 2.6 Ghz, 64 GB RAM, 20 MB L3, 256KB L2
- # Hypergraphs: [publicly available]
  - SuiteSparse Matrix Collection 184
  - SAT Competition 2014 (3 representations) 92·3
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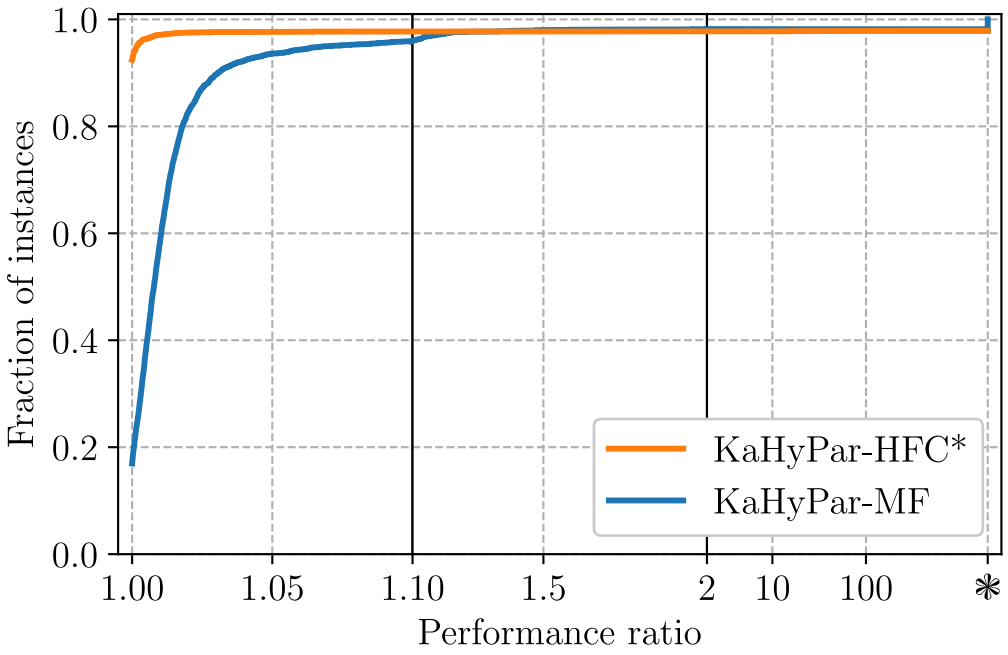
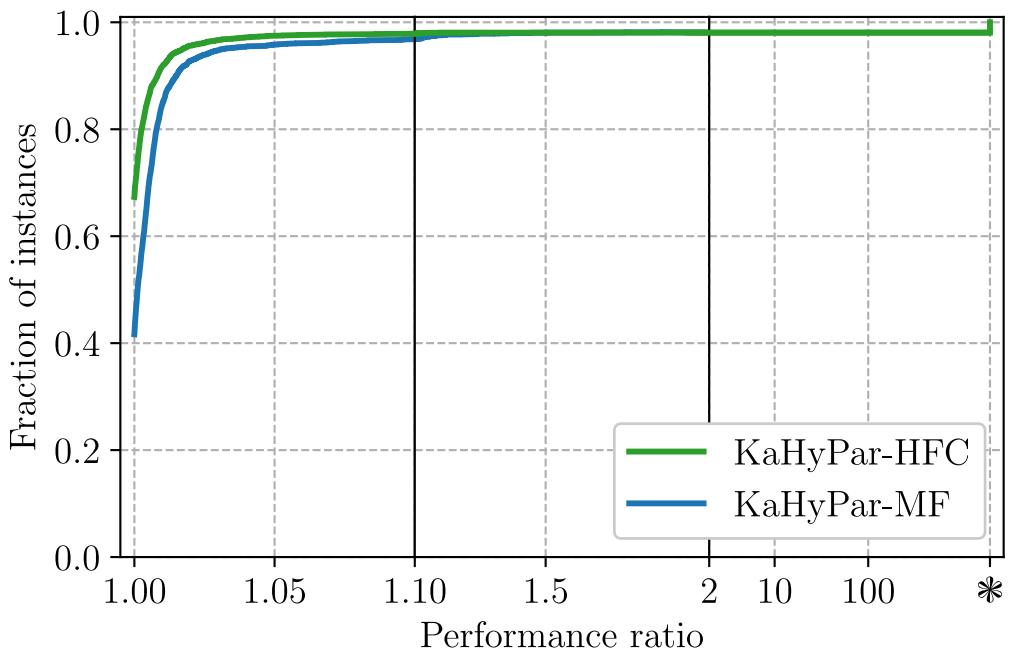
- Comparing **KaHyPar-HFC*** and **KaHyPar-HFC** with:
  - KaHyPar-MF
  - hMetis-R(ecursive Bisection) & hMetis-K (direct -kway)
  - PaToH-D(efault) & PaToH-Q(uality)
  - Mondriaan
  - Zoltan-AlgD
  - HYPE

Comparison with previous KaHyPar





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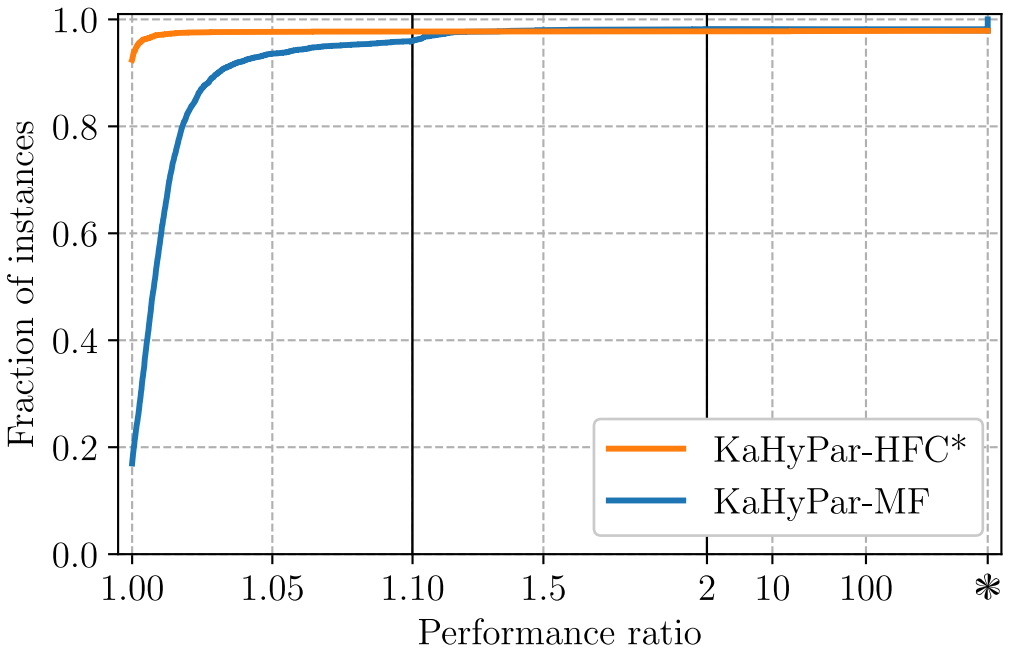
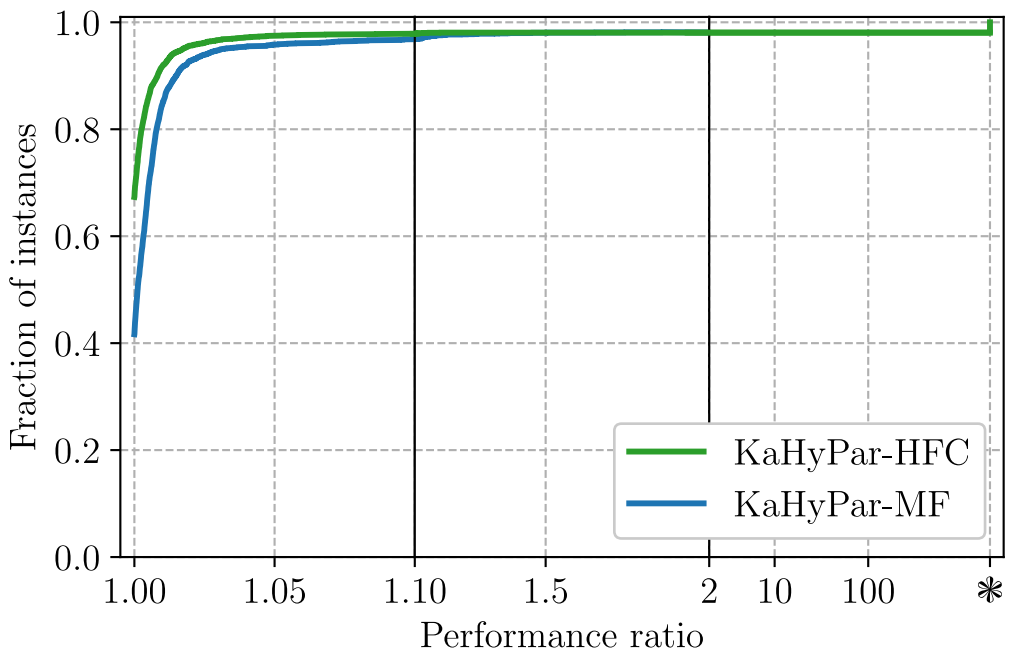
Comparison with previous KaHyPar





- performance ratio $r(a, i) = \frac{con(a,i)}{\min\{con(a',i) \mid a' \in \mathcal{A}\}}$ of algorithm a on instance i

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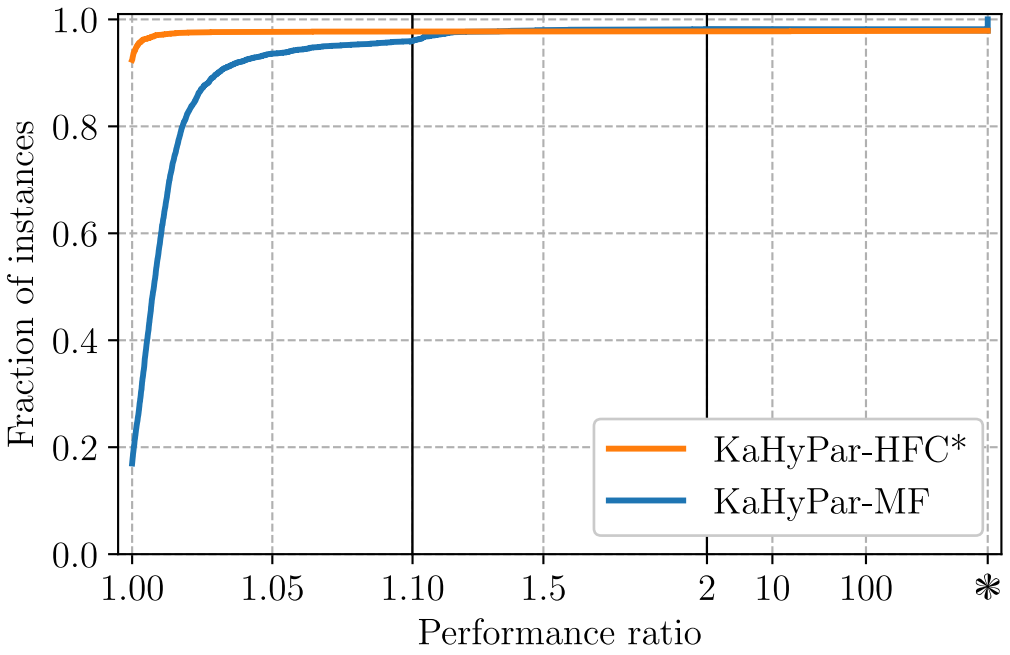
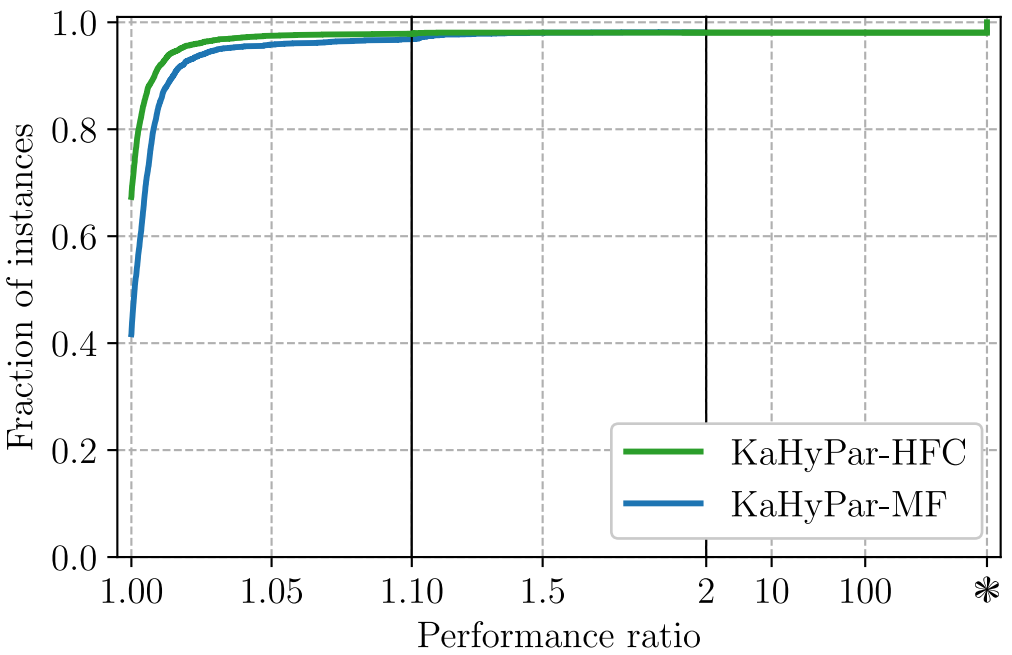
Comparison with previous KaHyPar





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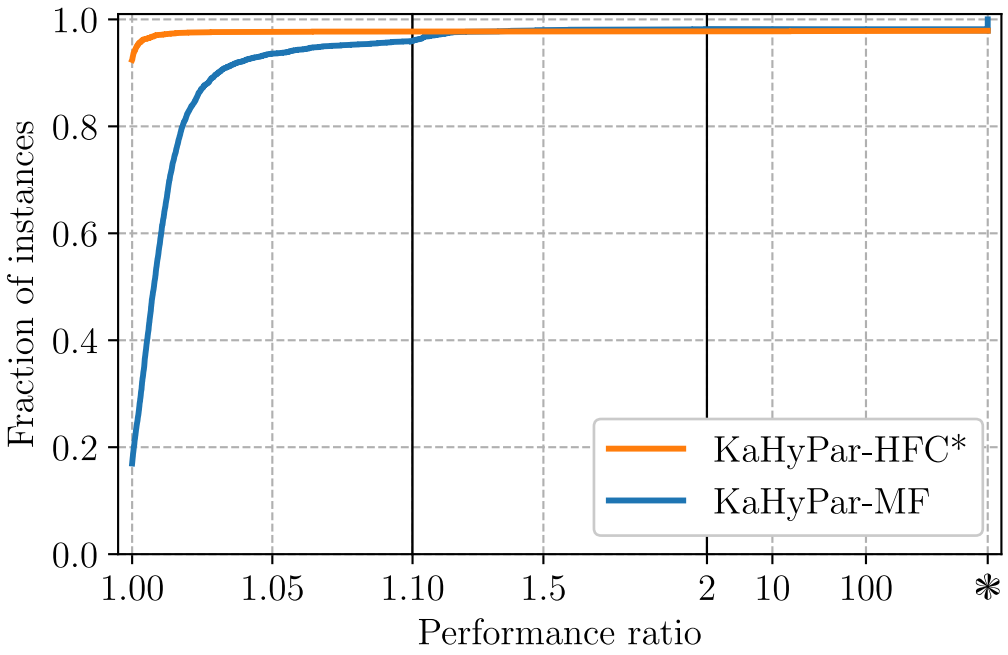

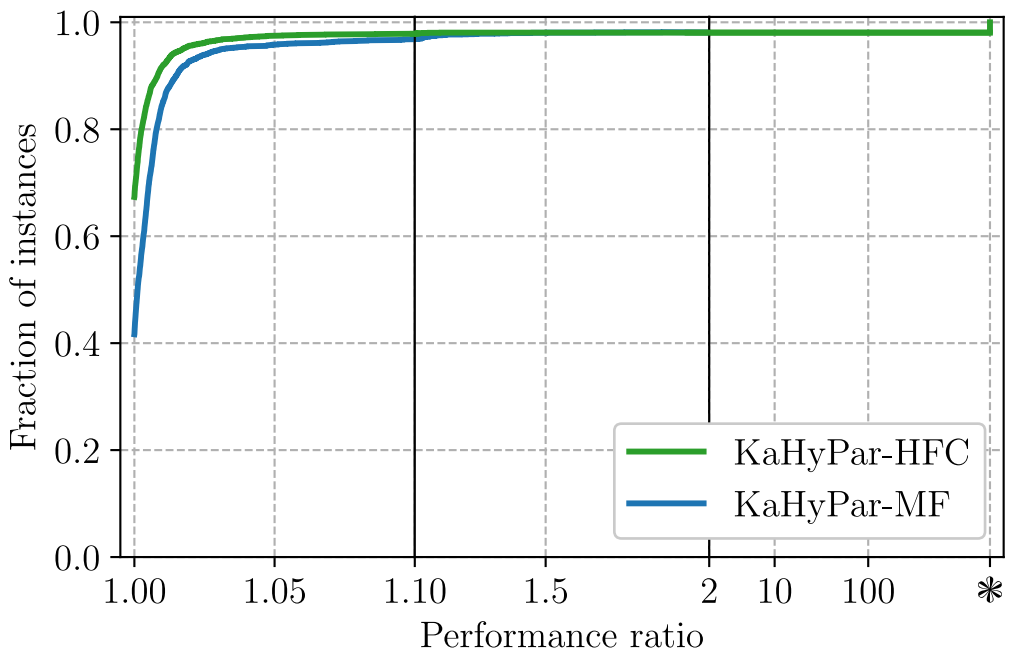

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- y-axis = fraction of instances with smaller performance ratio than value on x-axis
- x = 1 $\Rightarrow$ fraction of instances on which algorithm is the best
- higher is better

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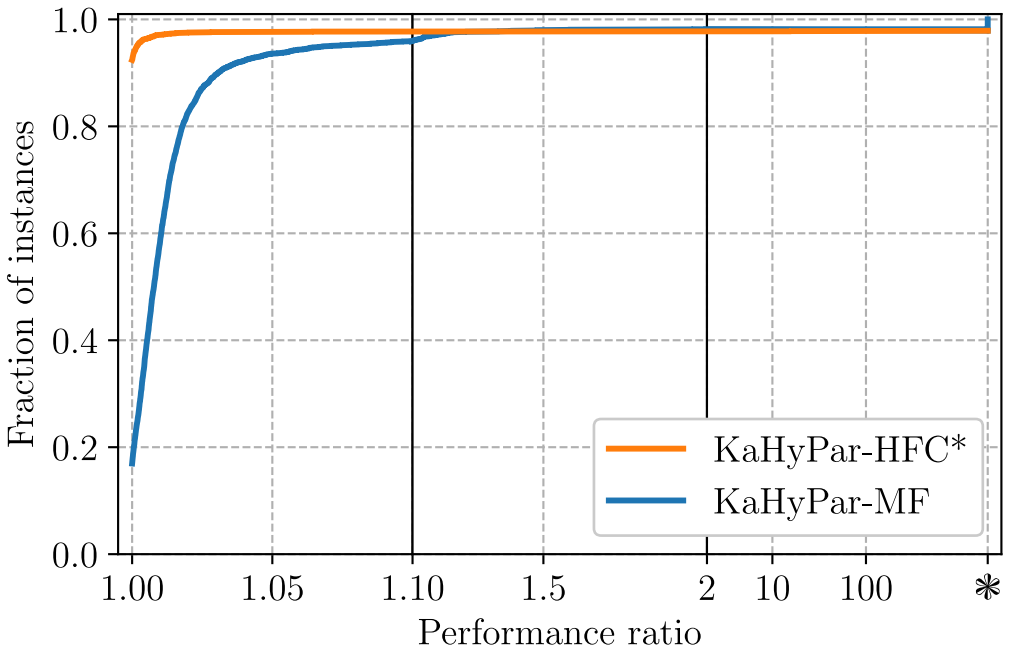
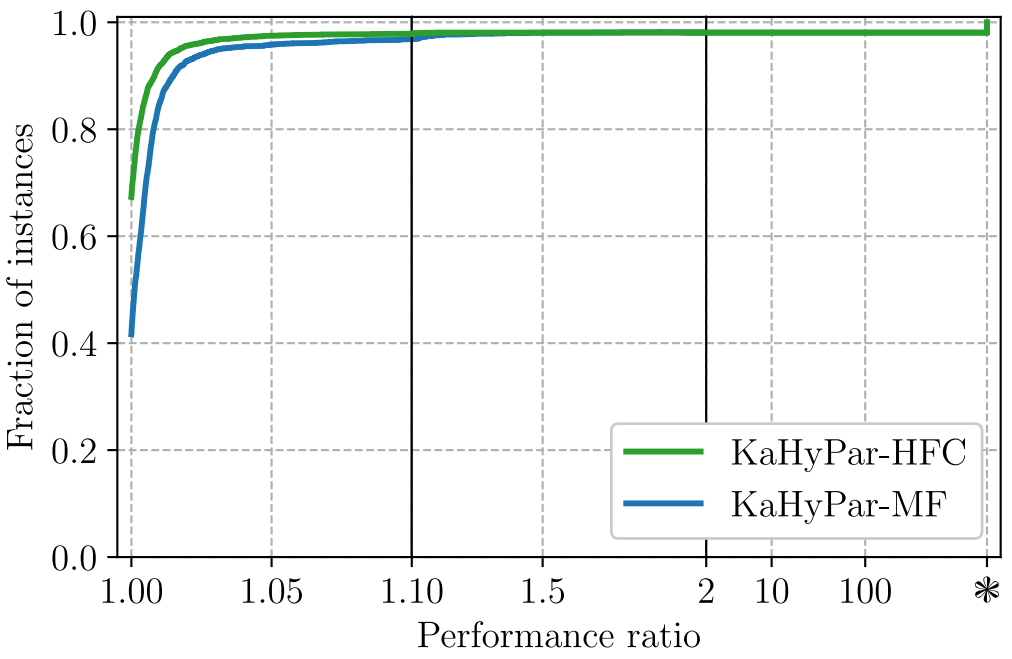
Comparison with previous KaHyPar





Algorithm	$t[s]$
KaHyPar-MF	67.07
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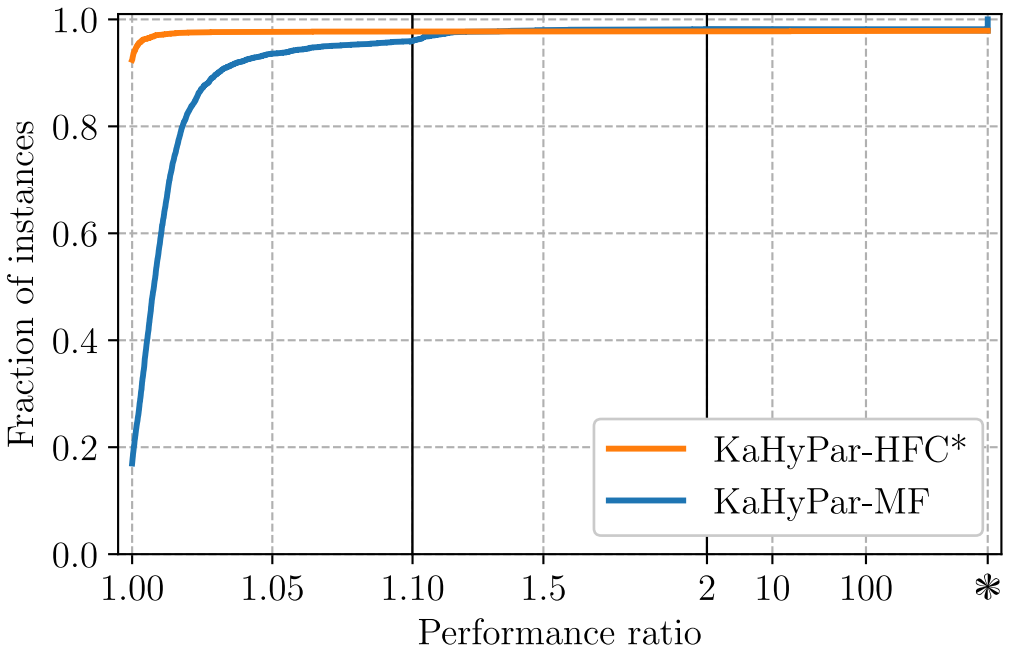

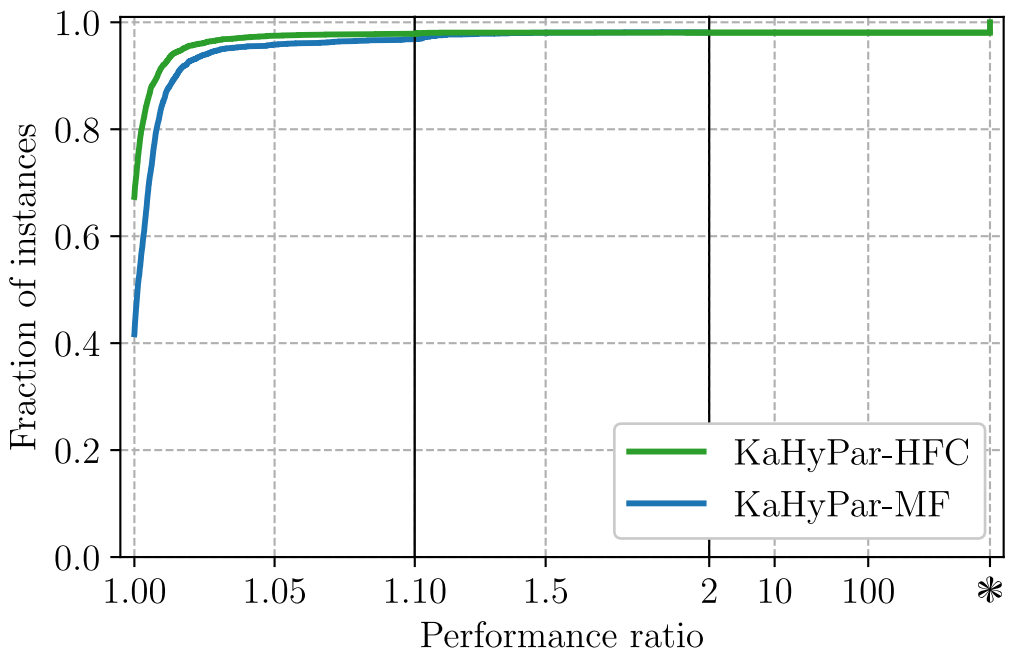

Algorithm	$t[s]$
KaHyPar-MF	67.07
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Algorithm	$t[s]$
KaHyPar-CA (no flows)	$\sim$ 40

take with a grain of salt!

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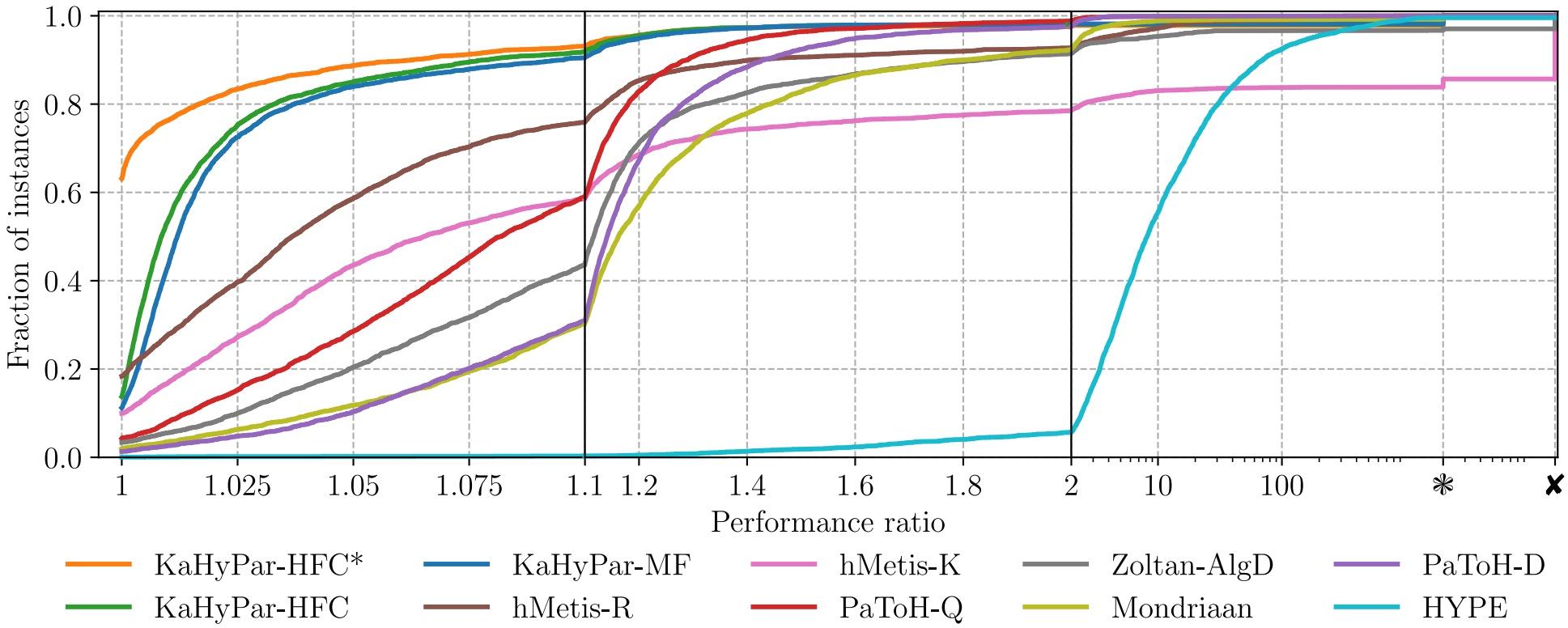
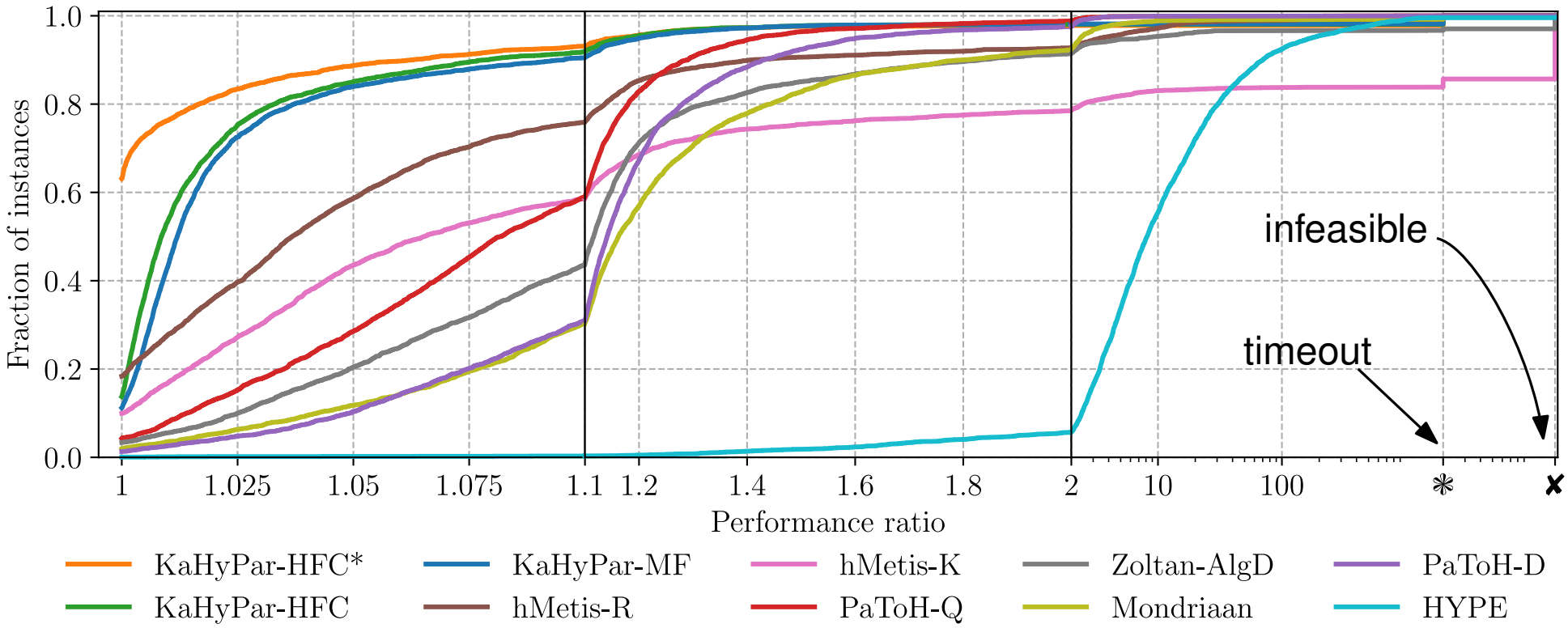
Comparison with other Partitioners



Comparison with other Partitioners



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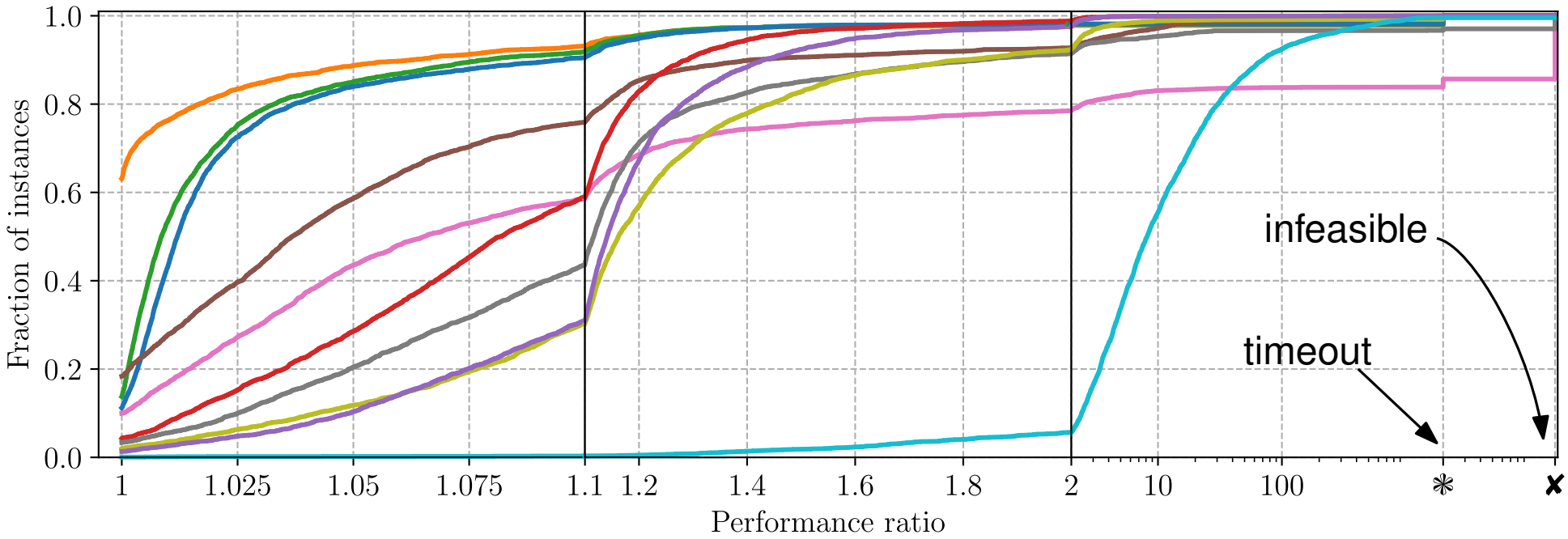
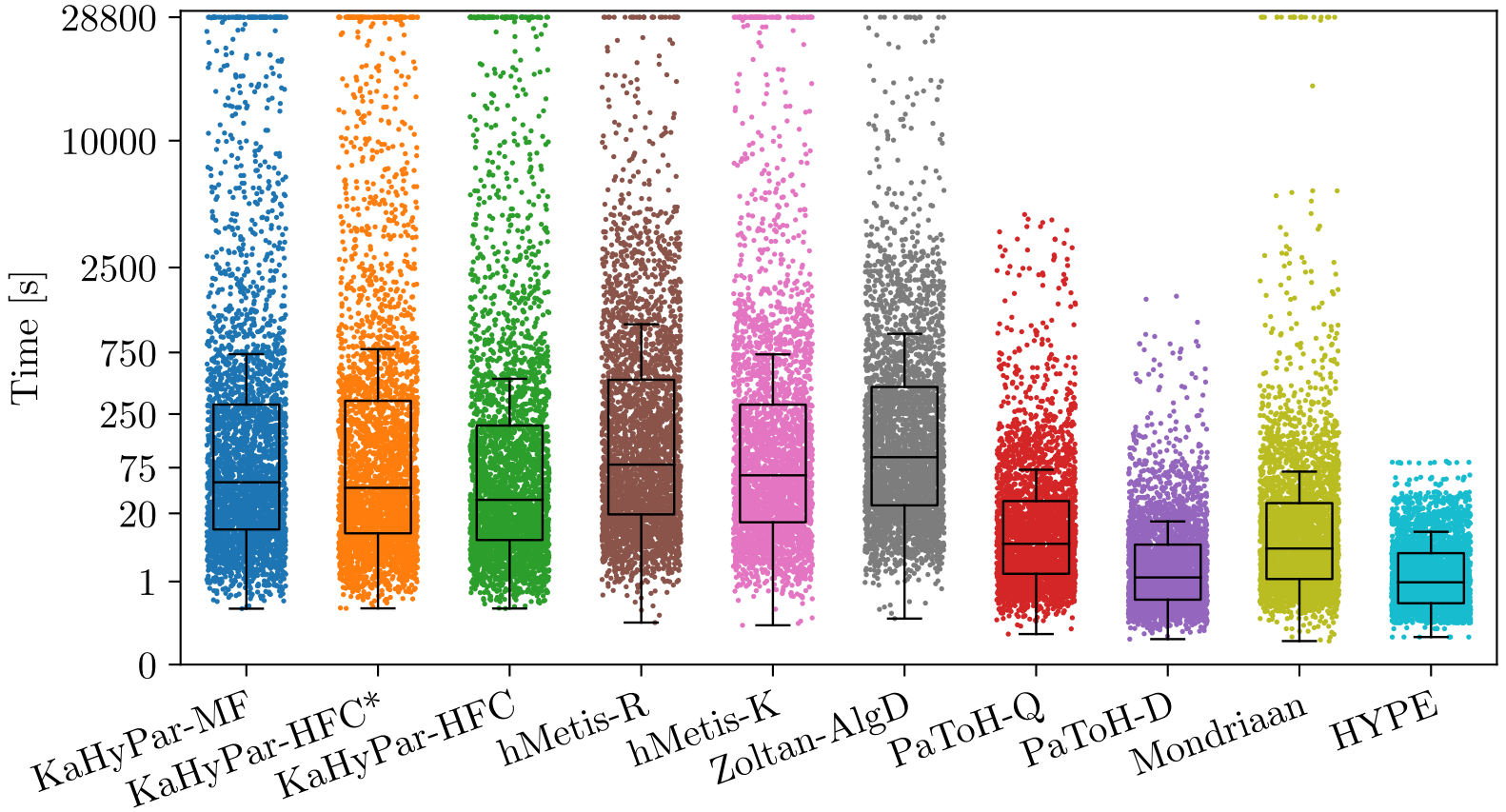


Algorithm	$t[s]$
KaHyPar-MF	67.07
KaHyPar-HFC*	62.49
KaHyPar-HFC	44.84
hMetis-R	96.55
hMetis-K	73.65

Algorithm	$t[s]$
Zoltan-AlgD	107.60
PaToH-Q	7.48
PaToH-D	1.47
Mondriaan	6.44
HYPE	1.03

Detailed Running Time



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Conclusion

KaHyPar-HFC – better and faster

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KaHyPar-HFC – better and faster

In the TR

- configuration study / assess impact of components
- different k / instance classes $\Rightarrow$ improves a lot on dual SAT and large k
- earlier balance with subset sum for special vertices
- distance-based piercing
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`https://kahypar.org`

`https://github.com/kahypar/kahypar`

`https://github.com/larsgottesbueren/WHFC`

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- parallel versions coming soon^{TM}

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